

## Unit 6

# Systems of linear differential equations



# Introduction

In Unit 2 you saw that the solution of the differential equation

$$\frac{dx}{dt} = Ax \quad (1)$$

(where  $A$  is a constant) is

$$x(t) = x_0 e^{At}. \quad (2)$$

Here  $x_0$  is an arbitrary constant. This unit generalises this differential equation and solution to the case of a system of several differential equations with more than one dependent variable. An example is the pair of differential equations

$$\frac{dx}{dt} = ax + by, \quad (3)$$

$$\frac{dy}{dt} = cx + dy, \quad (4)$$

where  $a, b, c$  and  $d$  are real constants. Here  $x(t)$  and  $y(t)$  are the dependent variables for which we want to find solutions. Such systems of differential equations arise frequently across all the mathematical sciences – in fact, whenever a system has constituent parts that interact with each other.

Because both of the unknown functions,  $x(t)$  and  $y(t)$ , occur in both of the differential equations, they must be solved ‘simultaneously’. At first sight this appears to be a difficult task. However, by using matrices it turns out that equations (3) and (4) can be cast in a form that is very similar to equation (1):

$$\begin{bmatrix} \dot{x} \\ \dot{y} \end{bmatrix} = \begin{bmatrix} a & b \\ c & d \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix},$$

which can then be written as

$$\dot{\mathbf{x}} = \mathbf{A}\mathbf{x}, \quad \text{where } \mathbf{x} = \begin{bmatrix} x \\ y \end{bmatrix} \text{ and } \mathbf{A} = \begin{bmatrix} a & b \\ c & d \end{bmatrix}. \quad (5)$$

This looks just like equation (1), and it would be satisfying to find a solution inspired by equation (2). An obvious extension to try is a solution of the form

$$\mathbf{x}(t) = \mathbf{v}e^{\lambda t}, \quad (6)$$

where  $\lambda$  is some constant scalar and  $\mathbf{v}$  is a constant vector. Substituting this into the left-hand side of equation (5), we get

$$\dot{\mathbf{x}} = \frac{d}{dt}(\mathbf{v}e^{\lambda t}) = \mathbf{v} \frac{d}{dt}e^{\lambda t} = \lambda \mathbf{v}e^{\lambda t}.$$

Substituting from equation (6) into the right-hand side of equation (5),

$\dot{\mathbf{x}} = \mathbf{A}\mathbf{x}$ , gives

$$\mathbf{A}\mathbf{x} = \mathbf{A}\mathbf{v}e^{\lambda t}.$$

Equating both sides of equation (5), we get

$$\mathbf{A}\mathbf{v} = \lambda \mathbf{v}.$$

Don't worry if you can't follow all the details here. This introduction is meant to be a sketch of what is to come. We cover the same material at a slower pace and in more depth in Section 1.

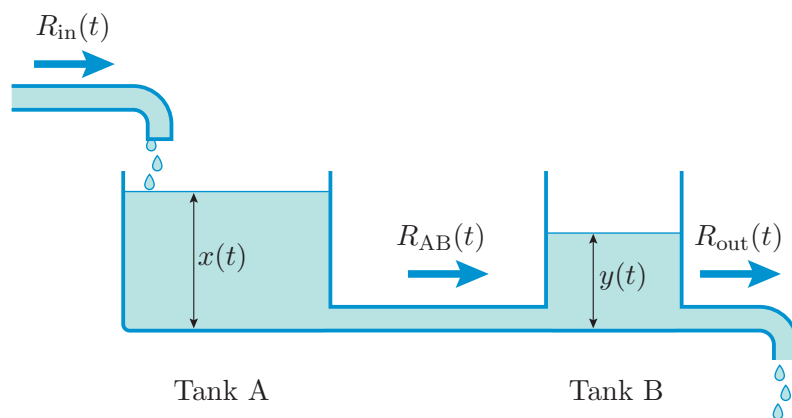
But this is the eigenvalue equation that we studied in Unit 5. So we see that equation (6) is a solution of equation (5) when  $\lambda$  and  $\mathbf{v}$  are an eigenvalue and an eigenvector of the matrix  $\mathbf{A}$ . In fact, this observation is the central message of this unit, and we highlight it because of its importance.

Putting systems of differential equations into matrix form and finding the eigenvalues and eigenvectors of the matrix of coefficients, is the key to solving systems of any number of linear differential equations of any order.

Before developing the mathematics further, let us first look at an example of how equations like (5) arise in a simple model. We will not give full details of the modelling process, since the aim is to provide motivation and to give a fairly rapid impression of where and how systems of differential equations might occur in practice. You should not spend too much time dwelling on the details, as we will never ask you to derive the differential equations for a system in any assessment.

## Fluid in tanks and pipes

Many engineering processes involve fluids being transferred in pipes from one tank to another. An example is shown in Figure 1. This could describe an industrial process where a liquid is pumped into a tank A, where it is treated in some way, before being transferred through a pipe to tank B, where it is stored before being supplied to run a machine.



**Figure 1** The depths of liquid in two tanks are  $x(t)$  and  $y(t)$

To describe the behaviour of this system we need two variables, namely the depth of the liquid in each tank. The depths of liquid at time  $t$  in tank A and tank B are  $x(t)$  and  $y(t)$ , respectively (measured in metres). We assume that liquid is poured into tank A at a rate of  $R_{\text{in}}(t)$  (measured in cubic metres per second). There is a flow of liquid from tank A to tank B at a rate  $R_{\text{AB}}(t)$ , and fluid is pumped out of tank B at a rate  $R_{\text{out}}(t)$ .

The rate of change of the depths is proportional to the rate at which fluid is added:

$$\dot{x} = k_A [R_{\text{in}}(t) - R_{\text{AB}}(t)], \quad (7)$$

$$\dot{y} = k_B [R_{\text{AB}}(t) - R_{\text{out}}(t)], \quad (8)$$

where  $k_A$  and  $k_B$  are two constants. We assume that the rate at which liquid flows through the pipe connecting tank A and tank B is proportional to the difference in height of the fluid in the two tanks, i.e. there is no pumping of fluid from tank A to tank B.

Thus we write

$$R_{\text{AB}} = K_{\text{AB}}(x - y), \quad (9)$$

where  $K_{\text{AB}}$  is a constant.

Substituting equation (9) into equations (7) and (8) gives the equations of motion for the heights of the fluid:

$$\dot{x} = -k_A K_{\text{AB}} x + k_A K_{\text{AB}} y + k_A R_{\text{in}}(t),$$

$$\dot{y} = k_B K_{\text{AB}} x - k_B K_{\text{AB}} y - k_B R_{\text{out}}(t).$$

These equations have the form

$$\dot{x} = ax + by + f(t), \quad (10)$$

$$\dot{y} = cx + dy + g(t). \quad (11)$$

Here  $a$ ,  $b$ ,  $c$  and  $d$  are constants, and  $f(t)$  and  $g(t)$  are functions of time describing the rate at which fluid is poured into the first tank and pumped out of the second tank, respectively:  $f(t) = k_A R_{\text{in}}(t)$ ,  $g(t) = -k_B R_{\text{out}}(t)$ .

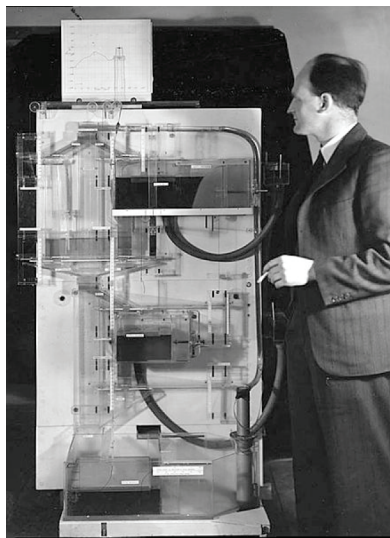
The values of the constants are  
 $a = -k_A K_{\text{AB}}$ ,  $b = -a$ ,  
 $c = k_B K_{\text{AB}}$ ,  $d = -c$ .

If we set  $f(t) = 0$  and  $g(t) = 0$ , so that no fluid is entering the first tank or being pumped from the second tank, then equations (10) and (11) are exactly the same as equations (3) and (4), with solution given by equation (6) in terms of the eigenvalues and eigenvectors of the matrix of

coefficients  $\mathbf{A} = \begin{bmatrix} a & b \\ c & d \end{bmatrix}$ . For  $f \neq 0$  and  $g \neq 0$ , a solution in terms of eigenvalues and eigenvectors can still be found, as we will see in the next section. In fact, the main job of Section 1 will be to find the general solution of systems of differential equations that have the same form as equations (10) and (11).

This application to the flow of fluid in tanks may seem somewhat specialised. However, all fluid flows obey the same physical laws, and systems of coupled linear differential equations find applications in many contexts involving fluid flow, such as understanding the drainage of rainwater by river systems.

The mathematical equations that describe the flow of fluids also apply to other physical phenomena such as the flow of current in electrical circuits, and non-physical phenomena such as models of the flow of money in the economy. The box below tells the story of an ingenious economist who exploited this duality to build an analogue computer to model the British economy.



**Figure 2** Bill Phillips with his MONIAC

### MONIAC: macroeconomics for plumbers!

The MONIAC (Monetary National Income Analogue Computer) was created in 1949 by electrical engineer turned economist Bill Phillips to model the economy of the United Kingdom (UK) – see Figure 2. Phillips was still a student at the London School of Economics when he created his first MONIAC in his landlady's garage in Croydon at a cost of £400.

The MONIAC was an analogue computer that used the flow of water to model the workings of an economy. It consisted of a series of transparent tanks and pipes. Each tank represented some aspect of the UK national economy, and the flow of money around the economy was illustrated by coloured water. At the top was a large tank called the treasury. Water (representing money) flowed from the treasury to other tanks representing the various ways in which the country could spend its money – for example, there were tanks for health and education. To increase spending on health, a tap could be opened to drain water from the treasury to the tank that represented health spending. Water then ran further down the model to other tanks, representing other interactions in the economy. Water could be pumped back to the treasury from some of the tanks to represent taxation. Changes in tax rates were modelled by increasing or decreasing pumping speeds. Import and export were represented by water draining from the model and additional water being poured into the model.

Phillips had realised that the set of differential equations that described the flow of money around the economy was the same as the set that described the flow of fluids between tanks. So by building the MONIAC, he was building a model of the economy.

The MONIAC had primarily been designed as a teaching aid, but was soon discovered also to be an effective economic simulator (accurate to  $\pm 2\%$ ), at a time when electronic digital computers that could run complex economic simulations were unavailable. A number of MONIAC machines were eventually built, ending up in companies, banks and universities around the world. One of the few remaining working machines is now on permanent display at the British Science Museum.

## Systems of coupled linear equations in a broader context

Equations like (3) and (4) are described as linear because the right-hand side involves only linear functions of  $x$  and  $y$ . However, most dynamical

systems are described by *non-linear* equations of the form

$$\begin{aligned}\dot{x} &= f(x, y), \\ \dot{y} &= g(x, y),\end{aligned}$$

where  $f(x, y)$  and  $g(x, y)$  are *any* functions of  $x$  and  $y$ . Although in general it is not possible to solve such equations analytically, it is possible to get a lot of information about how the solutions behave using analytical methods. In particular, the behaviour of the solution near *equilibrium points*, where  $\dot{x} = \dot{y} = 0$ , is studied using the techniques of this section. We will return to the discussion of non-linear dynamics in Unit 13.

There is another reason for studying systems of coupled linear equations. Quantum mechanics, which is our most fundamental theory of physical processes, can be expressed in terms of an equation of motion, similar to equation (5), for vectors in an abstract space of physical states.

### From coupled differential equations to images of brains

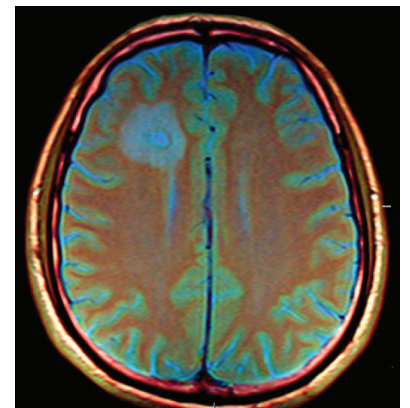
Particles like electrons and protons have a property called spin, which allows them to behave like tiny magnets when placed in a strong magnetic field. Hospitals use an invaluable imaging technique called magnetic resonance imaging (MRI) that is based on this idea.

Your body contains many hydrogen atoms distributed unevenly across your various tissues. At the heart of each hydrogen atom is a spinning proton that can be thought of as a tiny magnet. According to the general principles of quantum physics, the spin of such a proton is represented by a two-component column vector  $\mathbf{v}$ , called the *spin state vector*. The components of this vector are complex numbers, and they tell us about the orientation of the proton's spin at any given instant. The spin state vector may vary with time, satisfying an equation known as the *Schrödinger equation*, which takes the form

$$\frac{d\mathbf{v}}{dt} = \frac{1}{i\hbar} \mathbf{H}\mathbf{v}. \quad (12)$$

Here,  $\mathbf{H}$  is a  $2 \times 2$  matrix called the *Hamiltonian*,  $\mathbf{v}$  is a  $2 \times 1$  column vector (the spin state vector) and  $\hbar$  is a physical constant that appears throughout quantum physics. The only unusual feature of equation (12) is that it involves complex numbers: in general,  $\mathbf{H}$  and  $\mathbf{v}$  both have complex elements. Otherwise, equation (12) is of exactly the form that you have met before.

A concerted oscillation, in the directions of proton spins, is set off by the MRI scanner somewhere inside your body. This produces a tiny electromagnetic signal that can be detected outside. So understanding how to solve equation (12), and hence the motion of proton spins, provides the key to creating MRI images such as that shown in Figure 3.



**Figure 3** MRI image of the brain

## Study guide

In this unit we show you how to find the solution of certain systems of differential equations. We begin in the next section with systems where the derivatives are of first order, like those in equations (3) and (4), and equations (10) and (11). In Section 2 we consider the most commonly occurring systems where the derivatives are of second order. In the physical or engineering sciences, these systems describe the motion of objects that are coupled together in such a way that they have a vibrating or oscillating motion. In Section 3 we consider a special type of vibration, called ‘normal modes’, where the components of the system all vibrate with the same frequency.

This unit assumes a knowledge of first- and second-order differential equations, as covered in Units 2 and 3, and of eigenvalues and eigenvectors, as covered in Unit 5.

## 1 First-order systems

In this section we describe the techniques that you need in order to solve systems of first-order differential equations. Most of the examples and exercises concentrate on systems with only two (occasionally three) dependent variables, because the algebra is easier to understand. Once you have learned how to solve these, the extension of the technique to higher numbers of dependent variables is straightforward.

Subsection 1.1 describes how to write systems of differential equations in matrix form, and defines two types: homogeneous and inhomogeneous. Subsections 1.2 and 1.3 show you how to solve the homogeneous type. Subsections 1.4 and 1.5 show you how to solve the inhomogeneous type. The method parallels the methods that you saw in Unit 3 for solving second-order homogeneous and inhomogeneous differential equations of a single dependant variable.

### 1.1 Matrix notation

In Unit 5 you saw that any system of linear equations can be written in matrix form. For example, the equations

$$\begin{cases} 3x + 2y = 5, \\ x + 4y = 5, \end{cases}$$

can be written in matrix form as

$$\begin{bmatrix} 3 & 2 \\ 1 & 4 \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} 5 \\ 5 \end{bmatrix},$$

that is, as

$$\mathbf{Ax} = \mathbf{b}, \quad \text{where } \mathbf{A} = \begin{bmatrix} 3 & 2 \\ 1 & 4 \end{bmatrix}, \quad \mathbf{x} = \begin{bmatrix} x \\ y \end{bmatrix} \quad \text{and } \mathbf{b} = \begin{bmatrix} 5 \\ 5 \end{bmatrix}.$$

Note that sometimes we group a system of equations using a brace ( $\{$ ), as here, but often we do not.

In a similar way, we can write systems of linear differential equations in matrix form. To see what is involved, consider the system

$$\begin{cases} \dot{x} = 3x + 2y + 5t, \\ \dot{y} = x + 4y + 5, \end{cases} \quad (13)$$

where  $x$  and  $y$  are functions of  $t$ . This has the same form as equations (10) and (11) derived in the Introduction. It can be written in matrix form as

$$\begin{bmatrix} \dot{x} \\ \dot{y} \end{bmatrix} = \begin{bmatrix} 3 & 2 \\ 1 & 4 \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix} + \begin{bmatrix} 5t \\ 5 \end{bmatrix},$$

that is, as

$$\dot{\mathbf{x}} = \mathbf{A}\mathbf{x} + \mathbf{h}, \quad \text{where } \mathbf{A} = \begin{bmatrix} 3 & 2 \\ 1 & 4 \end{bmatrix}, \quad \mathbf{x} = \begin{bmatrix} x \\ y \end{bmatrix} \quad \text{and } \mathbf{h} = \begin{bmatrix} 5t \\ 5 \end{bmatrix}.$$

So we have converted a system of differential equations into a single matrix differential equation. Note that the components of  $\mathbf{x}$  are functions of  $t$ , and so generally are the components of  $\mathbf{h}$ . The matrix  $\mathbf{A}$  is independent of  $t$  and is called the **matrix of coefficients**.

We can similarly represent systems of three, or more, linear differential equations in matrix form. For example, the system

$$\begin{cases} \dot{x} = 3x + 2y + 2z + e^t, \\ \dot{y} = 2x + 2y + 2e^t, \\ \dot{z} = 2x + 4z, \end{cases}$$

can be written in matrix form as  $\dot{\mathbf{x}} = \mathbf{A}\mathbf{x} + \mathbf{h}$ , where

$$\mathbf{A} = \begin{bmatrix} 3 & 2 & 2 \\ 2 & 2 & 0 \\ 2 & 0 & 4 \end{bmatrix}, \quad \mathbf{x} = \begin{bmatrix} x \\ y \\ z \end{bmatrix} \quad \text{and} \quad \mathbf{h} = \begin{bmatrix} e^t \\ 2e^t \\ 0 \end{bmatrix} = \begin{bmatrix} 1 \\ 2 \\ 0 \end{bmatrix} e^t. \quad (14)$$

There are two types of matrix differential equation that you must be able to recognise. These are defined as follows.

### Definition

A matrix differential equation of the form  $\dot{\mathbf{x}} = \mathbf{A}\mathbf{x} + \mathbf{h}$  is said to be **homogeneous** if  $\mathbf{h} = \mathbf{0}$ , and **inhomogeneous** otherwise.

For example, the system

$$\begin{cases} \dot{x} = 2x + 3y, \\ \dot{y} = 2x + y, \end{cases} \quad (15) \quad \text{Here } x \text{ and } y \text{ are functions of } t.$$

has matrix form

$$\begin{bmatrix} \dot{x} \\ \dot{y} \end{bmatrix} = \begin{bmatrix} 2 & 3 \\ 2 & 1 \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix} \quad (16)$$

and so is homogeneous, whereas systems (13) and (14) are inhomogeneous.

Notice that we write the derivatives on the left-hand side. On the right-hand side we vertically align all the terms in  $x$ ,  $y$  and  $z$  separately, leaving a space where a term is zero.

Note that in an inhomogeneous system, some, *but not all*, of the components of  $\mathbf{h}$  may be 0.

**Exercise 1**

Write each of the following systems in matrix form, and classify it as homogeneous or inhomogeneous.

$$(a) \begin{cases} \dot{x} = 2x + y + 1 \\ \dot{y} = x - 2 \end{cases}$$

$$(b) \begin{cases} \dot{x} = y \\ \dot{y} = t \end{cases}$$

$$(c) \begin{cases} \dot{x} = 5x \\ \dot{y} = x + 2y + z \\ \dot{z} = x + y + 2z \end{cases}$$

**1.2 Homogeneous systems: the eigenvalue method****General solution**

We now show how to solve the first-order homogeneous case,  $\dot{\mathbf{x}} = \mathbf{A}\mathbf{x}$ . This was partially discussed in the Introduction, where we discovered that the method involves calculating eigenvalues and eigenvectors of matrices, which was covered in the previous unit.

Suppose that we are given a set of coupled, first-order, homogeneous, linear differential equations, like those in system (15). Then we put them in matrix form

$$\dot{\mathbf{x}} = \mathbf{A}\mathbf{x}, \quad (17)$$

as in equation (16). We assume that the matrix  $\mathbf{A}$  has coefficients that are constants, i.e. do not depend on the independent variable  $t$ . Then  $\mathbf{A}$  has eigenvectors  $\mathbf{v}$  and eigenvalues  $\lambda$  that are also independent of  $t$  and satisfy the eigenvalue equation

$$\mathbf{A}\mathbf{v} = \lambda\mathbf{v}. \quad (18)$$

In guessing this form for the solution, we are assuming that all components have the same exponential dependence on time.

Then it is easy to show that  $\mathbf{x}(t) = \mathbf{v}e^{\lambda t}$  is a solution of the differential equation. Substituting  $\mathbf{x}(t) = \mathbf{v}e^{\lambda t}$  into the left-hand side of equation (17), we get

$$\dot{\mathbf{x}} = \frac{d}{dt}(\mathbf{v}e^{\lambda t}) = \mathbf{v} \frac{d}{dt}e^{\lambda t} = \lambda\mathbf{v}e^{\lambda t}.$$

Substituting into the right-hand side of equation (17) and using equation (18), we get

$$\mathbf{A}\mathbf{x} = \mathbf{A}\mathbf{v}e^{\lambda t} = \lambda\mathbf{v}e^{\lambda t}.$$

So the left- and right-hand sides are equal, and we have the following result.

A system of differential equations

$$\dot{\mathbf{x}} = \mathbf{A}\mathbf{x}$$

has a solution given by

$$\mathbf{x} = \mathbf{v}e^{\lambda t},$$

where  $\lambda$  is an eigenvalue of the matrix  $\mathbf{A}$  corresponding to an eigenvector  $\mathbf{v}$ .

The question now arises as to how we find the *general solution*. The following example illustrates the idea for a pair of simultaneous differential equations.

### Example 1

- (a) Find two independent solutions of the simultaneous differential equations

$$\begin{cases} \dot{x} = x + 4y, \\ \dot{y} = x - 2y. \end{cases}$$

(Hint: The matrix  $\begin{bmatrix} 1 & 4 \\ 1 & -2 \end{bmatrix}$  has eigenvectors  $\begin{bmatrix} 4 \\ 1 \end{bmatrix}$  and  $\begin{bmatrix} 1 \\ -1 \end{bmatrix}$ , with corresponding eigenvalues 2 and  $-3$ .)

- (b) Find the general solution.

### Solution

- (a) The differential equations can be written in the matrix form

$$\begin{bmatrix} \dot{x} \\ \dot{y} \end{bmatrix} = \begin{bmatrix} 1 & 4 \\ 1 & -2 \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix}.$$

So the matrix of coefficients is

$$\mathbf{A} = \begin{bmatrix} 1 & 4 \\ 1 & -2 \end{bmatrix}.$$

Therefore using values for the eigenvalues and eigenvectors given in the hint, we can construct two independent solutions  $\mathbf{v}e^{\lambda t}$ :

$$\mathbf{x}_1 = \begin{bmatrix} 4 \\ 1 \end{bmatrix} e^{2t} \quad \text{and} \quad \mathbf{x}_2 = \begin{bmatrix} 1 \\ -1 \end{bmatrix} e^{-3t}.$$

- (b) So there are two independent solutions,  $\mathbf{x}_1(t)$  and  $\mathbf{x}_2(t)$ . It turns out, for reasons discussed below, that the general solution is a general linear combination of these, i.e.  $\alpha\mathbf{x}_1 + \beta\mathbf{x}_2$ . Hence the general solution is

$$\begin{bmatrix} x \\ y \end{bmatrix} = \alpha \begin{bmatrix} 4 \\ 1 \end{bmatrix} e^{2t} + \beta \begin{bmatrix} 1 \\ -1 \end{bmatrix} e^{-3t},$$

where  $\alpha$  and  $\beta$  are arbitrary constants. So the general solution in component form is

$$x = 4\alpha e^{2t} + \beta e^{-3t} \quad \text{and} \quad y = \alpha e^{2t} - \beta e^{-3t}.$$

**Important note** Note that we *cannot* change the equation for  $x$  to  $x = \alpha e^{2t} + \beta e^{-3t}$  (i.e. absorb the constant 4 into  $\alpha$ ) and leave the equation for  $y$  unchanged, since  $\alpha$  occurs in both the equation for  $x$  and the equation for  $y$ . This is an important difference between differential equations of a single variable and coupled differential equations.

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### Exercise 2

(a) Find two independent solutions of

$$\begin{cases} \dot{x} = 3x + 2y, \\ \dot{y} = x + 4y. \end{cases}$$

(Hint: The eigenvectors of  $\begin{bmatrix} 3 & 2 \\ 1 & 4 \end{bmatrix}$  are  $\begin{bmatrix} 1 \\ 1 \end{bmatrix}$  and  $\begin{bmatrix} -2 \\ 1 \end{bmatrix}$ , with corresponding eigenvalues 5 and 2.)

(b) Find the general solution.

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The reason why the linear combination of solutions is the general solution can be seen as follows. First, since  $\dot{\mathbf{x}}_1 = \mathbf{A}\mathbf{x}_1$  and  $\dot{\mathbf{x}}_2 = \mathbf{A}\mathbf{x}_2$ , the linear combination  $\mathbf{x} = \alpha\mathbf{x}_1 + \beta\mathbf{x}_2$  satisfies

$$\dot{\mathbf{x}} = \alpha\dot{\mathbf{x}}_1 + \beta\dot{\mathbf{x}}_2 = \alpha\mathbf{A}\mathbf{x}_1 + \beta\mathbf{A}\mathbf{x}_2 = \mathbf{A}(\alpha\mathbf{x}_1 + \beta\mathbf{x}_2) = \mathbf{A}\mathbf{x}.$$

This is a simple extension of the principle of superposition discussed in Unit 3.

So  $\mathbf{x} = \alpha\mathbf{x}_1 + \beta\mathbf{x}_2$  is a solution of the system of differential equations if  $\mathbf{x}_1$  and  $\mathbf{x}_2$  are. The fact that it is the *general solution* follows because  $\mathbf{x}$  contains two arbitrary constants ( $\alpha$  and  $\beta$ ), and this is sufficient for a system of two first-order differential equations.

In Example 1 we saw that a system of two differential equations has a  $2 \times 2$  matrix of coefficients, which gives rise to two independent solutions. The general solution is a general linear combination of these and so has two arbitrary constants ( $\alpha$  and  $\beta$ ). In the general case, a system of  $n$  differential equations has an  $n \times n$  matrix of coefficients, which gives rise to  $n$  independent solutions of the form  $\mathbf{v}e^{\lambda t}$ . The general solution is a general linear combination of these and contains  $n$  arbitrary constants.

You should note that the method outlined above works only when the  $n \times n$  matrix of coefficients has  $n$  (linearly independent) eigenvectors. The method fails if it has fewer. However, we do not consider such anomalous cases in this module.

Although this method works for both real and complex eigenvalues, in the next subsection we will find that it is convenient to modify it slightly in the complex case. So we use it in this form only for the real case. We summarise our method for real eigenvalues as follows.

### Procedure 1 Solution of a homogeneous system with real eigenvalues

To solve a system of linear constant-coefficient first-order differential equations  $\dot{\mathbf{x}} = \mathbf{A}\mathbf{x}$ , where  $\mathbf{A}$  is an  $n \times n$  matrix, do the following.

1. Find the eigenvalues  $\lambda_1, \lambda_2, \dots, \lambda_n$  and a corresponding set of eigenvectors  $\mathbf{v}_1, \mathbf{v}_2, \dots, \mathbf{v}_n$ .
2. Write down the general solution in the form

$$\mathbf{x} = C_1 \mathbf{v}_1 e^{\lambda_1 t} + C_2 \mathbf{v}_2 e^{\lambda_2 t} + \dots + C_n \mathbf{v}_n e^{\lambda_n t}, \quad (19)$$

where  $C_1, C_2, \dots, C_n$  are arbitrary constants.

The case of complex eigenvalues will be covered in Subsection 1.3. The case where  $\mathbf{A}$  does not have  $n$  eigenvectors is beyond the scope of this module.

The next example applies this procedure to find the general solution of a system of three differential equations.

### Example 2

Find the general solution of the system of differential equations

$$\begin{cases} \dot{x} = 3x + 2y + 2z, \\ \dot{y} = 2x + 2y, \\ \dot{z} = 2x + 4z. \end{cases}$$

(Hint: The matrix  $\begin{bmatrix} 3 & 2 & 2 \\ 2 & 2 & 0 \\ 2 & 0 & 4 \end{bmatrix}$  has eigenvectors  $\begin{bmatrix} 2 \\ 1 \\ 2 \end{bmatrix}$ ,  $\begin{bmatrix} 1 \\ 2 \\ -2 \end{bmatrix}$  and  $\begin{bmatrix} -2 \\ 2 \\ 1 \end{bmatrix}$ , with corresponding eigenvalues  $\lambda_1 = 6$ ,  $\lambda_2 = 3$  and  $\lambda_3 = 0$ .)

### Solution

The matrix of coefficients is

$$\mathbf{A} = \begin{bmatrix} 3 & 2 & 2 \\ 2 & 2 & 0 \\ 2 & 0 & 4 \end{bmatrix}.$$

Using the values for the eigenvalues and eigenvectors in the hint, the general solution is therefore

$$\begin{bmatrix} x \\ y \\ z \end{bmatrix} = \alpha \begin{bmatrix} 2 \\ 1 \\ 2 \end{bmatrix} e^{6t} + \beta \begin{bmatrix} 1 \\ 2 \\ -2 \end{bmatrix} e^{3t} + \gamma \begin{bmatrix} -2 \\ 2 \\ 1 \end{bmatrix},$$

where  $\alpha$ ,  $\beta$  and  $\gamma$  are arbitrary constants. So in component form this becomes

$$\begin{aligned} x &= 2\alpha e^{6t} + \beta e^{3t} - 2\gamma, \\ y &= \alpha e^{6t} + 2\beta e^{3t} + 2\gamma, \\ z &= 2\alpha e^{6t} - 2\beta e^{3t} + \gamma. \end{aligned}$$

Note that the last term on the right-hand side corresponds to the term in  $e^{\lambda_3 t} = e^{0t} = 1$ .

### Exercise 3

Find the general solution of

$$\begin{cases} \dot{x} = 5x + 2y, \\ \dot{y} = 2x + 5y. \end{cases}$$

(Hint: The eigenvectors of  $\begin{bmatrix} 5 & 2 \\ 2 & 5 \end{bmatrix}$  are  $\begin{bmatrix} 1 \\ 1 \end{bmatrix}$  and  $\begin{bmatrix} 1 \\ -1 \end{bmatrix}$ , with corresponding eigenvalues 7 and 3.)

### Initial conditions

Now that we know how to find the general solution of a system of differential equations, let us use this to find solutions that satisfy specific initial conditions. The basic procedure is the same as used in Units 2 and 3, i.e. use the initial conditions to find particular values of the arbitrary constants. The following example illustrates the idea.

### Example 3

A particle moves in the  $xy$ -plane in such a way that its position  $(x, y)$  at any time  $t$  satisfies the simultaneous differential equations

$$\begin{cases} \dot{x} = x + 4y, \\ \dot{y} = x - 2y. \end{cases}$$

Find the position  $(x, y)$  at time  $t$  if  $x(0) = 2$  and  $y(0) = 3$ .

### Solution

This system of differential equations was solved in Example 1. The general solution was found to be

$$\begin{bmatrix} x \\ y \end{bmatrix} = \alpha \begin{bmatrix} 4 \\ 1 \end{bmatrix} e^{2t} + \beta \begin{bmatrix} 1 \\ -1 \end{bmatrix} e^{-3t},$$

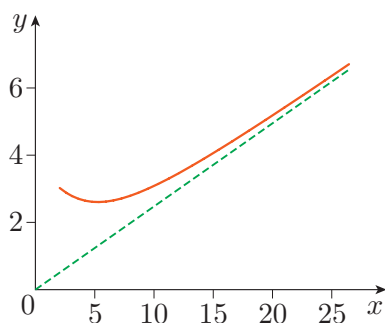
where  $\alpha$  and  $\beta$  are arbitrary constants.

Since  $x(0) = 2$  and  $y(0) = 3$ , we have, on putting  $t = 0$ ,

$$\begin{cases} 2 = 4\alpha + \beta, \\ 3 = \alpha - \beta. \end{cases}$$

Solving these equations gives  $\alpha = 1$ ,  $\beta = -2$ , so the required particular solution is

$$\begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} 4 \\ 1 \end{bmatrix} e^{2t} + \begin{bmatrix} -2 \\ 2 \end{bmatrix} e^{-3t}.$$



**Figure 4** Behaviour of the solution to Example 3. For  $t = 0$  the solution (orange line) starts at the point  $(2, 3)$ , but as  $t$  increases it approaches  $y = \frac{1}{4}x$  (green dashed line).

In the above example the particle starts at the point  $(2, 3)$  when  $t = 0$ , and follows a certain path as  $t$  increases. The ultimate direction of this path is easy to determine, because  $e^{-3t}$  is much smaller than  $e^{2t}$  when  $t$  is large, so we have  $\begin{bmatrix} x & y \end{bmatrix}^T \simeq \begin{bmatrix} 4 & 1 \end{bmatrix}^T e^{2t}$ , that is,  $x \simeq 4e^{2t}$  and  $y \simeq e^{2t}$ , so  $y \simeq \frac{1}{4}x$ . Thus the solution approaches the line  $y = \frac{1}{4}x$  as  $t$  increases. This behaviour is illustrated in Figure 4.

**Exercise 4**

(a) Use the above method to solve the system of differential equations

$$\begin{cases} \dot{x} = 5x + 2y, \\ \dot{y} = 2x + 5y, \end{cases}$$

given that  $x = 4$  and  $y = 0$  when  $t = 0$ .

(Hint: The result of Exercise 3 will be useful.)

(b) How does the solution behave for large  $t$ ?

**Exercise 5**

A particle moves in three-dimensional space in such a way that its position  $(x, y, z)$  at any time  $t$  satisfies the simultaneous differential equations

$$\begin{cases} \dot{x} = 5x, \\ \dot{y} = x + 2y + z, \\ \dot{z} = x + y + 2z. \end{cases}$$

Find the position  $(x, y, z)$  at time  $t$  if  $x(0) = 4$ ,  $y(0) = 6$  and  $z(0) = 0$ .

(Hint: The eigenvectors of  $\begin{bmatrix} 5 & 0 & 0 \\ 1 & 2 & 1 \\ 1 & 1 & 2 \end{bmatrix}$  are  $\begin{bmatrix} 2 \\ 1 \\ 1 \end{bmatrix}$ ,  $\begin{bmatrix} 0 \\ 1 \\ 1 \end{bmatrix}$  and  $\begin{bmatrix} 0 \\ 1 \\ -1 \end{bmatrix}$ , corresponding to eigenvalues 5, 3 and 1.)

In the next subsection we investigate what happens when the eigenvalues of the matrix  $\mathbf{A}$  are complex numbers.

**An alternative method**

There is an alternative method for solving systems of differential equations, which is occasionally used when the system is particularly simple. The idea is to eliminate all but one dependent variable. The technique can be illustrated with a pair of first-order equations such as

$$\begin{aligned} \dot{x} &= -2y, \\ \dot{y} &= 2x. \end{aligned}$$

Differentiating the first equation gives  $\ddot{x} = -2\dot{y}$ , so that  $\dot{y} = -\frac{1}{2}\ddot{x}$ . Then substituting this into the second equation gives

$$\ddot{x} = -4x.$$

So we have eliminated one of the variables, but obtained a differential equation of higher order, in this case a second-order differential equation that can be solved for  $x$  using the methods of Unit 3. However, using this technique on larger systems of differential equations is not practical, so we do not use it in this module.

### Solving systems of differential equations on a computer

In fact, the method described above is the opposite of what is commonly done when solving a higher-order differential equation on a computer. Given a second-order differential equation such as  $\ddot{y} + \dot{y} - 6y = 0$ , we would first define a new variable  $x = \dot{y}$ , then write the second-order equation as a pair of first-order equations:

$$\begin{aligned}\dot{x} &= -x + 6y, \\ \dot{y} &= x.\end{aligned}$$

Likewise, given a third-order differential equation such as  $\dddot{y} + 2\ddot{y} - 3\dot{y} + y = 0$ , we would define two new variables  $x = \dot{y}$  and  $z = \ddot{y} = \dot{x}$ , then write this as a system of three first-order equations:

$$\begin{aligned}\dot{x} &= z, \\ \dot{y} &= x, \\ \dot{z} &= 3x - y - 2z.\end{aligned}$$

These are then solved with one of the numerous computer packages that are designed specifically for solving systems of first-order differential equations.

## 1.3 Complex eigenvalues

This subsection deals with complex quantities. Before you begin, try the following warm-up exercise.

### Exercise 6

If  $z = 4 + 3i$ , show the following.

$$(a) \quad \bar{\bar{z}} = z \quad (b) \quad \operatorname{Re} z = \frac{1}{2}(z + \bar{z}) \quad (c) \quad \operatorname{Im} z = \frac{1}{2i}(z - \bar{z})$$

So far, all our examples and exercises have involved only *real* eigenvalues. We now investigate what happens when we have *complex* eigenvalues. In fact, because the arguments leading to Procedure 1 do not rely on the eigenvalues being real, they can also be used for the complex case. However, using equation (19) with complex eigenvalues  $\lambda_i$  means that the arbitrary constants  $C_1, C_2, \dots$  must also be complex for the solution  $\mathbf{x}$  to be real. It would be much more convenient if a real solution  $\mathbf{x}$  were expressed in terms of real quantities only. In this subsection we see how to modify equation (19) to a more useful form.

We begin with a simple example that leads to complex eigenvalues and illustrates the problem. Suppose that we want to solve the system of differential equations

$$\begin{cases} \dot{x} = y, \\ \dot{y} = -x. \end{cases} \quad (20)$$

We find the solution using Procedure 1. The matrix of coefficients is

$$\mathbf{A} = \begin{bmatrix} 0 & 1 \\ -1 & 0 \end{bmatrix},$$

and the eigenvalues of  $\mathbf{A}$  are easily shown to be

$$\begin{aligned} \lambda &= i \text{ with eigenvector } [1 \quad i]^T, \\ \lambda &= -i \text{ with eigenvector } [1 \quad -i]^T. \end{aligned}$$

Now since Procedure 1 works for complex as well as real eigenvalues, the general solution of the given system of differential equations can be written as

$$\begin{bmatrix} x \\ y \end{bmatrix} = C \begin{bmatrix} 1 \\ i \end{bmatrix} e^{it} + D \begin{bmatrix} 1 \\ -i \end{bmatrix} e^{-it}, \quad (21)$$

where  $C$  and  $D$  are arbitrary *complex* constants.

Since  $x$  and  $y$  are real, we would very much like the right-hand side of equation (21) to be written in terms of real quantities. In order to see how to do this, notice that the eigenvalues and eigenvectors  $\lambda$  and  $\mathbf{v}$  occur in complex conjugate pairs. (The complex conjugate of a vector  $\mathbf{v}$  is the vector  $\overline{\mathbf{v}}$  whose elements are the complex conjugates of the respective elements of  $\mathbf{v}$ . For example, if  $\mathbf{v} = [1 + 2i \quad -3i]^T$ , then  $\overline{\mathbf{v}} = [1 - 2i \quad 3i]^T$ .) This is always true for a matrix with real elements. So the solution will always be of the form

$$\mathbf{x} = C\mathbf{v}e^{\lambda t} + D\overline{\mathbf{v}}e^{\overline{\lambda}t}, \quad (22)$$

where  $C$  and  $D$  are arbitrary complex constants. This is the form of equation (21). Now, since  $\mathbf{x}$  is real, when we take the complex conjugate of both sides, we get

$$\mathbf{x} = \overline{\mathbf{x}} = \overline{C}\overline{\mathbf{v}}e^{\overline{\lambda}t} + \overline{D}\mathbf{v}e^{\lambda t}. \quad (23)$$

Here we have used the fact that  $\mathbf{x}$  and  $t$  are real, and  $z = \overline{\overline{z}}$  for any  $z$ . Comparing equations (22) and (23), we see that  $D = \overline{C}$  and hence

$$\mathbf{x} = C\mathbf{v}e^{\lambda t} + \overline{C}\overline{\mathbf{v}}e^{\overline{\lambda}t}. \quad (24)$$

This equation for  $\mathbf{x}$  is now manifestly real. To see this, simply take the complex conjugate of both sides and check that  $\mathbf{x} = \overline{\mathbf{x}}$ . Now let us set  $C = \alpha + i\beta$ , where  $\alpha$  and  $\beta$  are real. Then

$$\begin{aligned} \mathbf{x} &= (\alpha + i\beta)\mathbf{v}e^{\lambda t} + (\alpha - i\beta)\overline{\mathbf{v}}e^{\overline{\lambda}t} \\ &= \alpha(\mathbf{v}e^{\lambda t} + \overline{\mathbf{v}}e^{\overline{\lambda}t}) + i\beta(\mathbf{v}e^{\lambda t} - \overline{\mathbf{v}}e^{\overline{\lambda}t}). \end{aligned}$$

But for any complex number  $z$ ,  $\operatorname{Re} z = \frac{1}{2}(z + \overline{z})$  and  $\operatorname{Im} z = \frac{1}{2i}(z - \overline{z})$ , hence

$$\mathbf{x} = 2\alpha \operatorname{Re}(\mathbf{v}e^{\lambda t}) - 2\beta \operatorname{Im}(\mathbf{v}e^{\lambda t}).$$

Since  $C$  was an arbitrary complex constant,  $\alpha$  and  $\beta$  are arbitrary real constants, so we can absorb the factors of 2 and  $-2$  into them, giving

$$\mathbf{x} = \alpha \operatorname{Re}(\mathbf{v}e^{\lambda t}) + \beta \operatorname{Im}(\mathbf{v}e^{\lambda t}). \quad (25)$$

Notice that one eigenvalue is the complex conjugate of the other, and similarly for the eigenvectors.

The problem is analogous to the one in Unit 3 when we had complex roots of the auxiliary equation.

This is a simple form for  $\mathbf{x}$  that is obviously real and as such is sometimes called a **real-valued solution**. Let us apply it to the preceding example.

#### Example 4

For the system in equation (20), find a form of the solution  $\mathbf{x}$  that is clearly real.

#### Solution

The eigenvalues of the matrix of coefficients are

$$\lambda = i \text{ with eigenvector } \mathbf{v} = [1 \quad i]^T$$

and their complex conjugates

$$\bar{\lambda} = -i \text{ with eigenvector } \bar{\mathbf{v}} = [1 \quad -i]^T.$$

We are aiming to use equation (25), so let us choose

$$\mathbf{v}e^{\lambda t} = \begin{bmatrix} 1 \\ i \end{bmatrix} e^{it}.$$

We use Euler's formula,  $e^{it} = \cos t + i \sin t$ , to find the real and imaginary parts of  $\mathbf{v}e^{\lambda t}$ :

$$\begin{aligned} \mathbf{v}e^{\lambda t} &= \begin{bmatrix} 1 \\ i \end{bmatrix} (\cos t + i \sin t) \\ &= \underbrace{\begin{bmatrix} \cos t \\ -\sin t \end{bmatrix}}_{\text{real part}} + i \underbrace{\begin{bmatrix} \sin t \\ \cos t \end{bmatrix}}_{\text{imaginary part}}. \end{aligned}$$

We can now apply equation (25), to obtain

$$\mathbf{x} = \alpha \begin{bmatrix} \cos t \\ -\sin t \end{bmatrix} + \beta \begin{bmatrix} \sin t \\ \cos t \end{bmatrix},$$

where  $\alpha$  and  $\beta$  are arbitrary real constants.

For the case of a system of more than two differential equations, complex eigenvalues and eigenvectors also come in complex conjugate pairs – i.e. if  $\lambda, \mathbf{v}$  are eigenvalue and eigenvector, then so are  $\bar{\lambda}, \bar{\mathbf{v}}$ . So we generalise Procedure 1, incorporating equation (25), in the following way.

#### Procedure 2 Solution of a homogeneous system with complex eigenvalues

To obtain a real-valued solution of a system of linear constant-coefficient first-order differential equations  $\dot{\mathbf{x}} = \mathbf{A}\mathbf{x}$ , where  $\mathbf{A}$  is an  $n \times n$  matrix with distinct eigenvalues, some of which are complex (occurring in complex conjugate pairs  $\lambda$  and  $\bar{\lambda}$ , with corresponding complex conjugate eigenvectors  $\mathbf{v}$  and  $\bar{\mathbf{v}}$ ), do the following.

Euler's formula was used in a similar way in Unit 3, where it gave trigonometric solutions of second-order differential equations.

1. Find the eigenvalues  $\lambda_1, \lambda_2, \dots, \lambda_n$  and a corresponding set of eigenvectors  $\mathbf{v}_1, \mathbf{v}_2, \dots, \mathbf{v}_n$ .

2. Write down the general solution in the form

$$\mathbf{x} = C_1 \mathbf{v}_1 e^{\lambda_1 t} + C_2 \mathbf{v}_2 e^{\lambda_2 t} + \dots + C_n \mathbf{v}_n e^{\lambda_n t}.$$

3. Replace the complex terms  $\mathbf{v}e^{\lambda t}$  and  $\bar{\mathbf{v}}e^{\bar{\lambda}t}$  appearing in the general solution with  $\text{Re}(\mathbf{v}e^{\lambda t})$  and  $\text{Im}(\mathbf{v}e^{\lambda t})$ .

The general solution will then be real-valued for real  $C_1, C_2, \dots, C_n$ .

### Example 5

- (a) Find the general solution of the system of differential equations

$$\begin{cases} \dot{x} = 3x - y, \\ \dot{y} = 2x + y, \end{cases}$$

(Hint: The eigenvectors of  $\begin{bmatrix} 3 & -1 \\ 2 & 1 \end{bmatrix}$  are  $\mathbf{v} = \begin{bmatrix} 1 \\ 1-i \end{bmatrix}$  and  $\bar{\mathbf{v}} = \begin{bmatrix} 1 \\ 1+i \end{bmatrix}$ , with corresponding eigenvalues  $\lambda = 2+i$  and  $\bar{\lambda} = 2-i$ .)

- (b) Find the particular solution satisfying  $x = 3$  and  $y = 1$  when  $t = 0$ .

### Solution

- (a) The matrix of coefficients is

$$\mathbf{A} = \begin{bmatrix} 3 & -1 \\ 2 & 1 \end{bmatrix},$$

so using the hint, the general solution can be written as

$$\mathbf{x} = C \mathbf{v} e^{\lambda t} + D \bar{\mathbf{v}} e^{\bar{\lambda} t} = C \begin{bmatrix} 1 \\ 1-i \end{bmatrix} e^{(2+i)t} + D \begin{bmatrix} 1 \\ 1+i \end{bmatrix} e^{(2-i)t},$$

where  $C$  and  $D$  are arbitrary complex constants.

To obtain a real-valued solution, we follow Procedure 2 and write

$$\begin{aligned} \mathbf{v} e^{\lambda t} &= \begin{bmatrix} 1 \\ 1-i \end{bmatrix} e^{(2+i)t} \\ &= e^{2t} \begin{bmatrix} 1 \\ 1-i \end{bmatrix} e^{it} \\ &= e^{2t} \begin{bmatrix} 1 \\ 1-i \end{bmatrix} (\cos t + i \sin t) \\ &= e^{2t} \begin{bmatrix} \cos t + i \sin t \\ (1-i)(\cos t + i \sin t) \end{bmatrix} \\ &= e^{2t} \begin{bmatrix} \cos t + i \sin t \\ (\cos t + \sin t) + i(\sin t - \cos t) \end{bmatrix} \\ &= e^{2t} \underbrace{\begin{bmatrix} \cos t \\ \cos t + \sin t \end{bmatrix}}_{\text{real part}} + i e^{2t} \underbrace{\begin{bmatrix} \sin t \\ \sin t - \cos t \end{bmatrix}}_{\text{imaginary part}}. \end{aligned}$$

The real-valued general solution of the given system of equations is therefore

$$\begin{bmatrix} x \\ y \end{bmatrix} = \alpha e^{2t} \begin{bmatrix} \cos t \\ \cos t + \sin t \end{bmatrix} + \beta e^{2t} \begin{bmatrix} \sin t \\ \sin t - \cos t \end{bmatrix}, \quad (26)$$

where  $\alpha$  and  $\beta$  are arbitrary real constants.

- (b) In order to find the required particular solution, we substitute  $x = 3$ ,  $y = 1$  and  $t = 0$  into equation (26), to obtain

$$\begin{bmatrix} 3 \\ 1 \end{bmatrix} = \alpha \begin{bmatrix} 1 \\ 1 \end{bmatrix} + \beta \begin{bmatrix} 0 \\ -1 \end{bmatrix},$$

so  $3 = \alpha$  and  $1 = \alpha - \beta$ , giving  $\beta = 2$ , and the solution is therefore

$$\begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} (3 \cos t + 2 \sin t) \\ (\cos t + 5 \sin t) \end{bmatrix} e^{2t}.$$

### Exercise 7

- (a) Find the general solution of the system of differential equations

$$\begin{cases} \dot{x} = -3x - 2y, \\ \dot{y} = 4x + y. \end{cases}$$

(Hint: The eigenvectors of  $\begin{bmatrix} -3 & -2 \\ 4 & 1 \end{bmatrix}$  are  $\begin{bmatrix} 1 \\ -1 - i \end{bmatrix}$  and  $\begin{bmatrix} 1 \\ -1 + i \end{bmatrix}$ , with corresponding eigenvalues  $-1 + 2i$  and  $-1 - 2i$ .)

- (b) Find the particular solution satisfying  $x = y = 1$  when  $t = 0$ .

### Exercise 8

- (a) Find the general real-valued solution of the system of equations

$$\begin{cases} \dot{x} = x + z, \\ \dot{y} = x + 2y + z, \\ \dot{z} = -x + z. \end{cases}$$

(Hint: The eigenvectors of  $\begin{bmatrix} 1 & 0 & 1 \\ 1 & 2 & 1 \\ -1 & 0 & 1 \end{bmatrix}$  are  $\begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix}$ ,  $\begin{bmatrix} 1 \\ -i \\ i \end{bmatrix}$  and  $\begin{bmatrix} 1 \\ i \\ -i \end{bmatrix}$ , with corresponding eigenvalues  $2$ ,  $\lambda = 1 + i$  and  $\bar{\lambda} = 1 - i$ .)

- (b) Find the solution for which  $x = y = 1$  and  $z = 2$  when  $t = 0$ .

### Anomalous cases

The method outlined in the last two subsections works only when the  $n \times n$  matrix of coefficients has  $n$  (linearly independent) eigenvectors. As was mentioned in Unit 5, it can happen that an  $n \times n$  matrix has

fewer than  $n$  eigenvectors. In this case, when we construct the solution for  $\mathbf{x}$  given in Step 2 of Procedure 1, we find that it contains fewer than  $n$  arbitrary constants, so although it is a solution, it is not the *general* solution.

There are techniques for dealing with such anomalous cases, but they are not discussed in this module because they are not often employed in science. The reason for this is that in science, the systems of differential equations that we solve normally contain physical parameters – for example, the matrix of coefficients in the fluid example of the Introduction was formed from physical parameters like  $k_A K_{AB}$ . And for almost all values of these parameters, the  $n \times n$  matrix of coefficients will have  $n$  linearly independent eigenvectors. If there are particular values of the parameters where the matrix has fewer than  $n$  eigenvectors, then we can usually just alter them slightly to make the matrix well behaved again.

## 1.4 First-order inhomogeneous systems

In the previous subsections you saw how to solve a system of differential equations of the form  $\dot{\mathbf{x}} = \mathbf{A}\mathbf{x}$ , where  $\mathbf{A}$  is a given constant-coefficient matrix. We now extend our discussion to systems of the form

$$\dot{\mathbf{x}} = \mathbf{A}\mathbf{x} + \mathbf{h}(t),$$

where  $\mathbf{h}(t)$  is a given function of  $t$ . Our method involves finding a ‘particular integral’ for the system, and mirrors the approach that we took for inhomogeneous second-order differential equations in Unit 3.

Here we write  $\mathbf{h}(t)$  to emphasise that  $\mathbf{h}$  is a function of  $t$ . Henceforth we will abbreviate this to  $\mathbf{h}$ .

In Unit 3 we discussed inhomogeneous differential equations such as

$$\frac{d^2 y}{dx^2} + 9y = 2e^{3x}. \quad (27) \quad \text{See Unit 3, Example 9.}$$

To solve such an equation, we proceed as follows.

1. We first find the *complementary function* of the corresponding homogeneous equation

$$\frac{d^2 y}{dx^2} + 9y = 0,$$

which is, in this case,

$$y_c = C_1 \cos 3x + C_2 \sin 3x,$$

where  $C_1$  and  $C_2$  are arbitrary constants.

2. We then find a *particular integral* of the inhomogeneous equation (27). It is easy to check that

$$y_p = \frac{1}{9}e^{3x}$$

is such a particular integral.

The general solution  $y$  of the original equation is then obtained by adding these two functions to give

$$y = y_c + y_p = C_1 \cos 3x + C_2 \sin 3x + \frac{1}{9}e^{3x}.$$

A similar situation holds for systems of linear first-order differential equations. For example, in order to find the general solution of the inhomogeneous system

$$\begin{cases} \dot{x} = 3x + 2y + 4e^{3t}, \\ \dot{y} = x + 4y - e^{3t}, \end{cases} \quad (28)$$

which in matrix form becomes

$$\begin{bmatrix} \dot{x} \\ \dot{y} \end{bmatrix} = \begin{bmatrix} 3 & 2 \\ 1 & 4 \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix} + \begin{bmatrix} 4e^{3t} \\ -e^{3t} \end{bmatrix},$$

we first find the general solution of the corresponding homogeneous system

$$\begin{cases} \dot{x} = 3x + 2y, \\ \dot{y} = x + 4y, \end{cases}$$

which is the **complementary function**

$$\mathbf{x}_c = \begin{bmatrix} x_c \\ y_c \end{bmatrix} = \alpha \begin{bmatrix} 1 \\ 1 \end{bmatrix} e^{5t} + \beta \begin{bmatrix} -2 \\ 1 \end{bmatrix} e^{2t}, \quad (29)$$

where  $\alpha$  and  $\beta$  are arbitrary constants (see the solution to Exercise 2(b)).

We next find a particular solution, or **particular integral**, of the original inhomogeneous system (28). In Subsection 1.5 we will show that

$$\mathbf{x}_p = \begin{bmatrix} x_p \\ y_p \end{bmatrix} = \begin{bmatrix} 3 \\ -2 \end{bmatrix} e^{3t}, \quad (30)$$

is such a particular integral. The general solution of the original system (28) is then obtained by adding equations (29) and (30):

$$\begin{aligned} \begin{bmatrix} x \\ y \end{bmatrix} &= \begin{bmatrix} x_c \\ y_c \end{bmatrix} + \begin{bmatrix} x_p \\ y_p \end{bmatrix} \\ &= \alpha \begin{bmatrix} 1 \\ 1 \end{bmatrix} e^{5t} + \beta \begin{bmatrix} -2 \\ 1 \end{bmatrix} e^{2t} + \begin{bmatrix} 3 \\ -2 \end{bmatrix} e^{3t}. \end{aligned}$$

That this is the *general solution* can be seen as follows. Since  $\mathbf{x}_c$  is the general solution of the homogeneous equation  $\dot{\mathbf{x}}_c = \mathbf{A}\mathbf{x}_c$ , and  $\mathbf{x}_p$  is a particular integral of the inhomogeneous equation  $\dot{\mathbf{x}}_p = \mathbf{A}\mathbf{x}_p + \mathbf{h}$ , setting  $\mathbf{x} = \mathbf{x}_c + \mathbf{x}_p$  gives

$$\begin{aligned} \dot{\mathbf{x}} &= \dot{\mathbf{x}}_c + \dot{\mathbf{x}}_p = \mathbf{A}\mathbf{x}_c + \mathbf{A}\mathbf{x}_p + \mathbf{h} \\ &= \mathbf{A}(\mathbf{x}_c + \mathbf{x}_p) + \mathbf{h} \\ &= \mathbf{A}\mathbf{x} + \mathbf{h}. \end{aligned}$$

Therefore  $\mathbf{x}$  is a solution of the inhomogeneous equation and contains two arbitrary constants (from  $\mathbf{x}_c$ ). This is sufficient to guarantee that  $\mathbf{x}$  must be the general solution of the inhomogeneous equation. This argument is of course true for a system of any number of differential equations, so we have the following result.

This is  $\dot{\mathbf{x}} = \mathbf{A}\mathbf{x} + \mathbf{h}$ , where  $\mathbf{h} = [4e^{3t} \ -e^{3t}]^T$ .

We use the term *particular integral* rather than particular solution. The latter is more appropriately used for the solution of system (28) that satisfies given initial or boundary conditions.

### General solution of an inhomogeneous system

If  $\mathbf{x}_c$  is the complementary function of the homogeneous system  $\dot{\mathbf{x}} = \mathbf{A}\mathbf{x}$ , and  $\mathbf{x}_p$  is a particular integral of the system  $\dot{\mathbf{x}} = \mathbf{A}\mathbf{x} + \mathbf{h}$ , then  $\mathbf{x}_c + \mathbf{x}_p$  is the **general solution** of the system  $\dot{\mathbf{x}} = \mathbf{A}\mathbf{x} + \mathbf{h}$ .

#### Exercise 9

Write down the general solution of the system

$$\begin{cases} \dot{x} = 3x + 2y + t, \\ \dot{y} = x + 4y + 7t, \end{cases}$$

given that a particular integral is

$$x_p = t + \frac{4}{5}, \quad y_p = -2t - \frac{7}{10}.$$

(Hint: The eigenvectors of  $\begin{bmatrix} 3 & 2 \\ 1 & 4 \end{bmatrix}$  are  $\begin{bmatrix} 1 \\ 1 \end{bmatrix}$  and  $\begin{bmatrix} -2 \\ 1 \end{bmatrix}$ , with corresponding eigenvalues 5 and 2.)

Although it is easy to verify that equation (30) is a solution of the given system, by direct substitution, we now show you how to *determine* it.

## 1.5 Finding particular integrals

We now show you how to find a particular integral  $\mathbf{x}_p$  in some special cases. We consider the system  $\dot{\mathbf{x}} = \mathbf{A}\mathbf{x} + \mathbf{h}$  in the situations where  $\mathbf{h}$  is a vector whose components are:

- polynomial functions
- exponential functions.

Our treatment will be similar to that in Unit 3, where we found particular integrals for linear second-order differential equations using the method of undetermined coefficients. To illustrate the ideas involved, we consider the system

$$\dot{\mathbf{x}} = \mathbf{A}\mathbf{x} + \mathbf{h}, \quad \text{where } \mathbf{A} = \begin{bmatrix} 3 & 2 \\ 1 & 4 \end{bmatrix}.$$

The first stage in solving any inhomogeneous system is to find the complementary function, that is, the solution of the system  $\dot{\mathbf{x}} = \mathbf{A}\mathbf{x}$ . The complementary function for this system was found in the solution to Exercise 2(b):

$$\begin{bmatrix} x_c \\ y_c \end{bmatrix} = \alpha \begin{bmatrix} 1 \\ 1 \end{bmatrix} e^{5t} + \beta \begin{bmatrix} -2 \\ 1 \end{bmatrix} e^{2t}. \quad (31)$$

To this complementary function we add a particular integral that depends on the form of  $\mathbf{h}$ . We now look at examples of the above two forms for  $\mathbf{h}$ , and derive a particular integral in each case.

Here  $\mathbf{h} = [t \quad 7t]^T$ , so  $\mathbf{h}$  is linear in  $t$ .

You may have been tempted to use a simpler trial solution, of the form

$$\begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} at \\ ct \end{bmatrix}.$$

Unfortunately, this does not work – try it and see! You may recall something similar in Unit 3.

These equations hold for *all* values of  $t$ , which means that each of the bracketed terms must be zero.

### Example 6

Find the general solution of the system

$$\begin{cases} \dot{x} = 3x + 2y + t, \\ \dot{y} = x + 4y + 7t. \end{cases}$$

### Solution

The complementary function is given in equation (31).

We note that  $\mathbf{h}$  consists entirely of linear functions, so it seems natural to seek a particular integral of the form

$$\begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} at + b \\ ct + d \end{bmatrix},$$

where  $a, b, c$  and  $d$  are constants that we need to determine. So  $x = at + b$ ,  $y = ct + d$ , and differentiating these we get  $\dot{x} = a$ ,  $\dot{y} = c$ . Substituting these values into the simultaneous equations gives

$$\begin{cases} a = 3(at + b) + 2(ct + d) + t, \\ c = (at + b) + 4(ct + d) + 7t. \end{cases}$$

We now rearrange these equations, separating constant terms from the terms linear in  $t$ :

$$\begin{cases} (3a + 2c + 1)t + (3b + 2d - a) = 0, \\ (a + 4c + 7)t + (b + 4d - c) = 0. \end{cases} \quad (32)$$

Equating the coefficients of  $t$  to zero in equations (32) gives

$$\begin{cases} 3a + 2c + 1 = 0, \\ a + 4c + 7 = 0, \end{cases}$$

which have the solution

$$a = 1, \quad c = -2.$$

Equating the constant terms to zero in equations (32), and putting  $a = 1$ ,  $c = -2$ , gives the equations

$$\begin{cases} 3b + 2d - 1 = 0, \\ b + 4d + 2 = 0, \end{cases}$$

which have the solution

$$b = \frac{4}{5}, \quad d = -\frac{7}{10}.$$

Thus the required particular integral is

$$\begin{bmatrix} x_p \\ y_p \end{bmatrix} = \begin{bmatrix} t + \frac{4}{5} \\ -2t - \frac{7}{10} \end{bmatrix},$$

and the general solution is

$$\begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} x_c \\ y_c \end{bmatrix} + \begin{bmatrix} x_p \\ y_p \end{bmatrix} = \alpha \begin{bmatrix} 1 \\ 1 \end{bmatrix} e^{5t} + \beta \begin{bmatrix} -2 \\ 1 \end{bmatrix} e^{2t} + \begin{bmatrix} t + \frac{4}{5} \\ -2t - \frac{7}{10} \end{bmatrix}.$$

**Exercise 10**

Find the general solution of the system

$$\begin{cases} \dot{x} = x + 4y - t + 2, \\ \dot{y} = x - 2y + 5t. \end{cases}$$

(Hint: For the complementary function, see Example 1.)

**Example 7**

Find the general solution of the system

$$\begin{cases} \dot{x} = 3x + 2y + 4e^{3t}, \\ \dot{y} = x + 4y - e^{3t}. \end{cases}$$

Here  $\mathbf{h} = [4e^{3t} \quad -e^{3t}]^T$ , so  $\mathbf{h}$  is exponential.

**Solution**

The complementary function is given in equation (31). We note that both components of  $\mathbf{h}$  include the same exponential function  $e^{3t}$ , so it seems natural to seek a particular integral of the form

$$\begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} ae^{3t} \\ be^{3t} \end{bmatrix} = \begin{bmatrix} a \\ b \end{bmatrix} e^{3t},$$

where  $a$  and  $b$  are constants that we need to determine. So  $x = ae^{3t}$ ,  $y = be^{3t}$ , and differentiating these gives  $\dot{x} = 3ae^{3t}$ ,  $\dot{y} = 3be^{3t}$ . Substituting these values into the simultaneous equations gives

$$\begin{cases} 3ae^{3t} = 3ae^{3t} + 2be^{3t} + 4e^{3t}, \\ 3be^{3t} = ae^{3t} + 4be^{3t} - e^{3t}, \end{cases}$$

or, on dividing by  $e^{3t}$ ,

$$\begin{cases} 3a = 3a + 2b + 4, \\ 3b = a + 4b - 1. \end{cases}$$

Rearranging these equations gives

$$\begin{cases} 2b = -4, \\ a + b = 1, \end{cases}$$

which have the solution

$$a = 3, \quad b = -2.$$

Thus the required particular integral is

$$\begin{bmatrix} x_p \\ y_p \end{bmatrix} = \begin{bmatrix} 3e^{3t} \\ -2e^{3t} \end{bmatrix} = \begin{bmatrix} 3 \\ -2 \end{bmatrix} e^{3t},$$

and the general solution is

$$\begin{aligned} \begin{bmatrix} x \\ y \end{bmatrix} &= \begin{bmatrix} x_c \\ y_c \end{bmatrix} + \begin{bmatrix} x_p \\ y_p \end{bmatrix} \\ &= \alpha \begin{bmatrix} 1 \\ 1 \end{bmatrix} e^{5t} + \beta \begin{bmatrix} -2 \\ 1 \end{bmatrix} e^{2t} + \begin{bmatrix} 3 \\ -2 \end{bmatrix} e^{3t}. \end{aligned}$$

**Exercise 11**

Find the general solution of the system

$$\begin{cases} \dot{x} = x + 4y + 4e^{-t}, \\ \dot{y} = x - 2y + 5e^{-t}. \end{cases}$$

(*Hint*: The complementary function is the same as that of Exercise 10.)

We summarise our results in the following procedure.

**Procedure 3 Finding particular integrals**

To find a particular integral  $\mathbf{x}_p = [x_p \ y_p]^T$  for the system  $\dot{\mathbf{x}} = \mathbf{A}\mathbf{x} + \mathbf{h}$ , do the following.

1. When the elements of  $\mathbf{h}$  are polynomials of degree  $k$  or less, choose  $x_p$  and  $y_p$  to be polynomials of degree  $k$ .
2. When the elements of  $\mathbf{h}$  are multiples of the same exponential function, choose  $x_p$  and  $y_p$  to be multiples of this exponential function.

To determine the coefficients in  $x_p$  and  $y_p$ , substitute into the system of differential equations and equate coefficients if necessary.

**Exercise 12**

Consider the system of differential equations

$$\begin{cases} \dot{x} = 2x + 3y + e^{2t}, \\ \dot{y} = 2x + y + 4e^{2t}. \end{cases}$$

- (a) Evaluate the eigenvalues and eigenvectors of the matrix of coefficients.
- (b) Find the solution of this system subject to the initial conditions  $x(0) = \frac{5}{6}$ ,  $y(0) = \frac{2}{3}$ .
- (c) How does the solution found in part (b) behave for large  $t$ ?

**Other cases**

This short subsection is included for completeness, but **the material in it will not be assessed**.

### Combinations of cases

Procedure 3 allows you to determine the particular integral when the inhomogeneous term  $\mathbf{h}$  has components that are simple functions like polynomials or exponentials. When  $\mathbf{h}$  is a linear combination of these simple functions, for example

$$\mathbf{h}(t) = \mathbf{h}_1(t) + \mathbf{h}_2(t), \quad \text{where } \mathbf{h}_1 = \begin{bmatrix} 4 \\ -1 \end{bmatrix} e^{3t} \text{ and } \mathbf{h}_2 = \begin{bmatrix} 1 \\ 7 \end{bmatrix} t,$$

we find a particular integral for each of  $\mathbf{h}_1$  and  $\mathbf{h}_2$  separately, using the above method (see Examples 6 and 7), then add the two particular integrals together. We would use a similar trick if  $\mathbf{h}_1$  and  $\mathbf{h}_2$  were two exponential terms with different exponents.

### Exceptional cases

Occasionally Procedure 3 will fail to find a particular integral of an inhomogeneous system  $\dot{\mathbf{x}} = \mathbf{A}\mathbf{x} + \mathbf{h}$ . This usually occurs when the inhomogeneous part  $\mathbf{h}$  is related to the complementary function  $\mathbf{x}_c$ . For example,

$$\begin{cases} \dot{x} = 3x + 2y + 6e^{2t}, \\ \dot{y} = x + 4y + 3e^{2t}, \end{cases}$$

has a complementary function given by equation (31):

$$\begin{bmatrix} x_c \\ y_c \end{bmatrix} = \alpha \begin{bmatrix} 1 \\ 1 \end{bmatrix} e^{5t} + \beta \begin{bmatrix} -2 \\ 1 \end{bmatrix} e^{2t}.$$

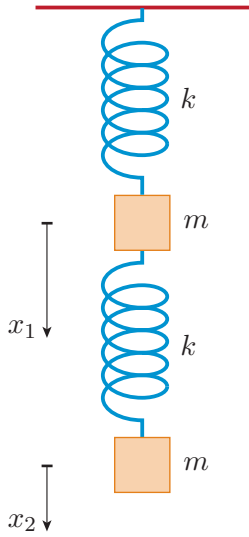
We see that the term  $e^{2t}$  occurs in both the complementary function and the inhomogeneous part  $\mathbf{h}$ . If we follow Procedure 3 and try to find a particular integral of the form  $\mathbf{x}_p = [a_1 \ a_2]^T e^{2t}$ , then we will find that the system of equations for  $a_1$  and  $a_2$  has no solution and so the method fails.

The resolution is identical to that discussed in Unit 3 when the method of undetermined coefficients failed for the same reason: we simply try a more general form for the particular integral. In this case we would try a solution of the form

$$\mathbf{x}_p = \begin{bmatrix} a_1 + b_1 t \\ a_2 + b_2 t \end{bmatrix} e^{2t}.$$

This is analogous to what was done in Unit 3 for second-order differential equations.

## 2 Second-order systems



**Figure 5** A compound harmonic oscillator; we analyse the vertical motion of these two masses, connected by springs

Forces and accelerations are normally written as vectors. However, when we have motion in only one direction, we just consider the vertical component as we have done here.

Don't worry if you can't follow this argument; it is not important for what follows.

In this section we show how the methods already introduced to solve systems of first-order differential equations can be adapted to systems of second-order differential equations. We restrict ourselves to considering homogeneous second-order systems, as the inhomogeneous case can be handled using the same technique as in the previous section. We begin with a short motivational section describing how the motion of bodies coupled by springs leads naturally to such equations. Don't worry too much about the details, as you will not be assessed on deriving equations of motion.

### 2.1 Mechanical oscillations and normal modes

In Unit 3 we discussed the motion of a mass suspended on a spring. We showed that this system exhibits a sinusoidal motion called simple harmonic motion. Here we consider a generalisation of this mass and spring system, illustrated in Figure 5. The system consists of two particles, each of mass  $m$ , suspended by springs with spring constant  $k$ , with one mass suspended below the other.

This system can rest in equilibrium with both masses stationary. In order to describe the motion of this system, we need to describe the displacement of the masses from this equilibrium position. Let the downward displacement of the upper mass from its equilibrium point be  $x_1$ , and let the downward displacement of the lower mass from its equilibrium point be  $x_2$ . We need to find equations of motion for  $x_1(t)$  and  $x_2(t)$ . These are obtained from Newton's second law:

$$m \frac{d^2 x_1}{dt^2} = F_1, \quad m \frac{d^2 x_2}{dt^2} = F_2, \quad (33)$$

where  $F_1$  is the force on the upper mass due to its displacement from its equilibrium position, and  $F_2$  is the force on the lower mass. These forces are linear in  $x_1$  and  $x_2$ , and can be shown to satisfy

$$\begin{aligned} F_1 &= k(x_2 - x_1) - kx_1 = k(x_2 - 2x_1), \\ F_2 &= k(x_1 - x_2). \end{aligned}$$

We do not expect you to be able to derive such equations. However, the following is an argument that makes them plausible.

Look at Figure 5. Suppose that we move the upper mass down by a positive amount  $\Delta$  and hold the lower mass still, so that  $x_1 = \Delta > 0$  and  $x_2 = 0$ . Then we expect an upwards restoring force from both springs on the upper mass, and a downwards restoring force from just the lower spring on the lower mass.

The equations for the forces give  $F_1 = k(x_2 - 2x_1) = -2k\Delta$  and  $F_2 = k(x_1 - x_2) = k\Delta$ . So  $F_1$  is upwards,  $F_2$  is downwards, and  $|F_1|$  is twice  $|F_2|$ , which is what we expect by intuition.

Now suppose that we move the lower mass down by a positive amount  $\Delta$  and hold the upper mass still, so that  $x_1 = 0$  and  $x_2 = \Delta > 0$ . Then we expect a downwards restoring force on the upper mass, and an equal and opposite upwards restoring force on the lower mass.

The equations for the forces give  $F_1 = k(x_2 - 2x_1) = k\Delta$  and  $F_2 = k(x_1 - x_2) = -k\Delta$ , which is what we expect by intuition.

Substituting these equations for the forces into equations (33), we obtain equations of motion for the pair of masses:

$$\begin{aligned} m\ddot{x}_1 &= k(x_2 - 2x_1), \\ m\ddot{x}_2 &= k(x_1 - x_2). \end{aligned}$$

This can be written in matrix form as

$$\begin{bmatrix} \ddot{x}_1 \\ \ddot{x}_2 \end{bmatrix} = \frac{k}{m} \begin{bmatrix} -2 & 1 \\ 1 & -1 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix}, \quad (34)$$

or as

$$\ddot{\mathbf{x}} = \mathbf{A}\mathbf{x}, \quad \text{where } \mathbf{x} = \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} \text{ and } \mathbf{A} = \frac{k}{m} \begin{bmatrix} -2 & 1 \\ 1 & -1 \end{bmatrix}. \quad (35)$$

This is similar in form to equation (5) of the Introduction, except that it involves second derivatives. In this section and the next we will see that there are solutions, derived from the eigenvalues and eigenvectors of  $\mathbf{A}$  and analogous to equation (6), in the form of a constant vector multiplying a function of time.

Among all the possible motions of this system there is a special motion, where the two masses oscillate with the same angular frequency  $\omega$ , but with possibly different amplitudes  $a_1$  and  $a_2$ . In this case the function of time is a sinusoidal function, and the solution typically has the form

$$\begin{bmatrix} x_1(t) \\ x_2(t) \end{bmatrix} = \sin(\omega t + \phi) \begin{bmatrix} a_1 \\ a_2 \end{bmatrix},$$

where  $\phi$  is a constant. These special solutions are called the *normal modes* of oscillation of the system and will be discussed in Section 3.

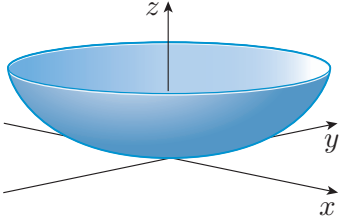
### Dynamics of other physical systems near equilibrium

Second-order equations of the form  $\ddot{\mathbf{x}} = \mathbf{A}\mathbf{x}$  find applications in many other physical situations. When we make a small displacement  $x$  of any system from equilibrium, the restoring forces can be expanded as a Taylor series in  $x$ :

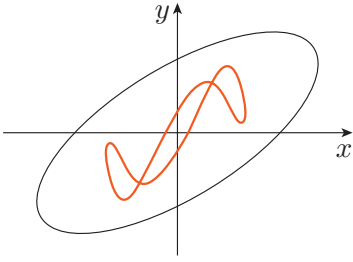
$$F(x) = Cx + Dx^2 + \dots,$$

where  $C$  and  $D$  are constants.

Note that the constant term in the series is equal to zero because the force is zero at the equilibrium:  $F(0) = 0$ .



**Figure 6** A bowl



**Figure 7** Typical path of a ball in an elliptical bowl, looked at from above

When the displacement  $x$  is small, the quadratic and higher-order terms can be neglected, so that  $F \simeq Cx$  and hence using Newton's second law we get

$$\ddot{x} = Ax,$$

where  $A = C/m$ , and  $m$  represents the mass of the system. If two or more coordinates  $(x_1, x_2, \dots)$  are required to describe the displacement from equilibrium, then the equation of motion for the small displacements takes a form similar to equation (35):

$$\ddot{\mathbf{x}} = \mathbf{A}\mathbf{x}, \quad (36)$$

where  $\mathbf{A}$  is a matrix and  $\mathbf{x} = [x_1 \ x_2 \ \dots]^T$  is a vector formed by the displacements. This equation describes the dynamics of most physical systems near equilibrium.

A simple example of this is when a ball bearing (a small metal ball) is placed in a bowl, like that in Figure 6, and set in motion. In that case,  $\mathbf{x}$  represents the vector displacement of the ball from the lowest part of the bowl, when looked at from above. And when  $|\mathbf{x}|$  is not too large, the motion of the ball can be described by equation (36), where the coefficients of the matrix  $\mathbf{A}$  are determined from the shape of the bottom of the bowl. Figure 7 illustrates a typical path that you might see if you looked down onto the surface of the bowl from above.

Let us now move on to the topic of how to solve systems of second-order differential equations.

## 2.2 Solving second-order systems

We will consider systems of linear constant-coefficient second-order differential equations of the form

$$\ddot{\mathbf{x}} = \mathbf{A}\mathbf{x}. \quad (37)$$

We will try to solve this equation in a similar manner to the way in which we solved the first-order case  $\dot{\mathbf{x}} = \mathbf{A}\mathbf{x}$  in the previous section.

Let  $\mathbf{v}$  be an eigenvector of the matrix  $\mathbf{A}$  with eigenvalue  $\lambda$ , so  $\mathbf{A}\mathbf{v} = \lambda\mathbf{v}$ . For the moment we will assume  $\lambda \neq 0$  (the case  $\lambda = 0$  will be covered shortly). Let us try a solution of the form  $\mathbf{x}(t) = \mathbf{v}e^{\mu t}$  for some number  $\mu$ . Substituting into the left- and right-hand sides of equation (37), we get

$$\frac{d^2}{dt^2}\mathbf{v}e^{\mu t} = \mathbf{A}\mathbf{v}e^{\mu t},$$

so

$$\mu^2\mathbf{v}e^{\mu t} = \lambda\mathbf{v}e^{\mu t}.$$

Thus  $\mu^2 = \lambda$ , hence  $\mathbf{x}(t) = \mathbf{v}e^{\sqrt{\lambda}t}$  and  $\mathbf{x}(t) = \mathbf{v}e^{-\sqrt{\lambda}t}$  are both solutions. So we have the following result.

For a system of differential equations

$$\ddot{\mathbf{x}} = \mathbf{A}\mathbf{x},$$

solutions are given by

$$\mathbf{x} = \mathbf{v}e^{\pm\sqrt{\lambda}t},$$

where  $\lambda$  is an eigenvalue of the matrix  $\mathbf{A}$  corresponding to an eigenvector  $\mathbf{v}$ .

The *general solution* is, as you may already have guessed, a linear combination of the solutions  $\mathbf{v}e^{+\sqrt{\lambda}t}$  and  $\mathbf{v}e^{-\sqrt{\lambda}t}$  for all eigenvalues  $\lambda$ . We must consider two cases:  $\lambda > 0$  and  $\lambda < 0$ . Let us consider the positive case first; the following example illustrates how to construct the general solution for a pair of simultaneous differential equations.

### Example 8

Find the general solution of the system of differential equations

$$\begin{cases} \ddot{x} = 3x + 2y, \\ \ddot{y} = x + 4y. \end{cases}$$

(*Hint:* The eigenvectors of  $\begin{bmatrix} 3 & 2 \\ 1 & 4 \end{bmatrix}$  are  $\mathbf{v}_1 = \begin{bmatrix} 1 \\ 1 \end{bmatrix}$  and  $\mathbf{v}_2 = \begin{bmatrix} -2 \\ 1 \end{bmatrix}$ , with corresponding eigenvalues  $\lambda_1 = 5$  and  $\lambda_2 = 2$ .)

### Solution

The matrix of coefficients is

$$\mathbf{A} = \begin{bmatrix} 3 & 2 \\ 1 & 4 \end{bmatrix}.$$

Using the hint, it follows that  $\mathbf{v}_1 e^{\pm\sqrt{\lambda_1}t} = \begin{bmatrix} 1 & 1 \end{bmatrix}^T e^{\pm\sqrt{5}t}$  and  $\mathbf{v}_2 e^{\pm\sqrt{\lambda_2}t} = \begin{bmatrix} -2 & 1 \end{bmatrix}^T e^{\pm\sqrt{2}t}$  are solutions. Hence the general solution is a linear combination of these:

$$\begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} 1 \\ 1 \end{bmatrix} (C_1 e^{\sqrt{5}t} + C_2 e^{-\sqrt{5}t}) + \begin{bmatrix} -2 \\ 1 \end{bmatrix} (C_3 e^{\sqrt{2}t} + C_4 e^{-\sqrt{2}t}),$$

where  $C_1, C_2, C_3$  and  $C_4$  are arbitrary constants.

In this example, we see that two second-order differential equations give rise to a general solution with four arbitrary constants – four terms arise from the positive and negative square roots of each eigenvalue. Not surprisingly, a system of  $n$  second-order differential equations gives rise to a general solution with  $2n$  arbitrary constants.

**Exercise 13**

Find the general solution of the system of differential equations

$$\begin{cases} \ddot{x} = 5x + 2y, \\ \ddot{y} = 2x + 5y. \end{cases}$$

(Hint: The eigenvectors of  $\begin{bmatrix} 5 & 2 \\ 2 & 5 \end{bmatrix}$  are  $\begin{bmatrix} 1 \\ 1 \end{bmatrix}$  and  $\begin{bmatrix} 1 \\ -1 \end{bmatrix}$ , with corresponding eigenvalues 7 and 3.)

All this is fine when the eigenvalues are positive. However, if an eigenvalue  $\lambda$  is negative, then  $\sqrt{\lambda} = i\sqrt{-\lambda}$  is pure imaginary, and we have solutions of the form

$$\mathbf{x} = C_1 \mathbf{v} e^{i\sqrt{-\lambda}t} + C_2 \mathbf{v} e^{-i\sqrt{-\lambda}t} + \text{other eigenvalue terms.}$$

Hence the constants  $C_1$  and  $C_2$  must be complex for  $\mathbf{x}$  to be real. A similar problem occurred in Subsection 1.3, and as there, we use Euler's formula to manipulate our solution into one involving sines and cosines. So (ignoring the other eigenvalue terms) we have

$$\begin{aligned} \mathbf{x} &= C_1 \mathbf{v} (\cos(\sqrt{-\lambda}t) + i \sin(\sqrt{-\lambda}t)) + C_2 \mathbf{v} (\cos(\sqrt{-\lambda}t) - i \sin(\sqrt{-\lambda}t)) \\ &= (C_1 + C_2) \mathbf{v} \cos(\sqrt{-\lambda}t) + i(C_1 - C_2) \mathbf{v} \sin(\sqrt{-\lambda}t) \\ &= \alpha \mathbf{v} \cos(\sqrt{-\lambda}t) + \beta \mathbf{v} \sin(\sqrt{-\lambda}t), \end{aligned}$$

where  $\alpha = C_1 + C_2$  and  $\beta = i(C_1 - C_2)$ . Since  $C_1$  and  $C_2$  are arbitrary, so are  $\alpha$  and  $\beta$ . Furthermore, if  $\mathbf{x}$  is real, then clearly  $\alpha$  and  $\beta$  must be real too. We summarise this as follows.

If  $\lambda$  is a *negative* eigenvalue of the matrix  $\mathbf{A}$  corresponding to an eigenvector  $\mathbf{v}$ , then

$$\mathbf{x} = \mathbf{v} \cos(\sqrt{-\lambda}t) \quad \text{and} \quad \mathbf{x} = \mathbf{v} \sin(\sqrt{-\lambda}t)$$

are solutions of the system of differential equations  $\ddot{\mathbf{x}} = \mathbf{A}\mathbf{x}$ .

**Example 9**

Find the general solution of the system of differential equations

$$\begin{cases} \ddot{x} = x + 4y, \\ \ddot{y} = x - 2y. \end{cases}$$

**Solution**

The matrix of coefficients is

$$\mathbf{A} = \begin{bmatrix} 1 & 4 \\ 1 & -2 \end{bmatrix}.$$

The eigenvectors of  $\mathbf{A}$  can be shown to be

$$\mathbf{v}_1 = [4 \quad 1]^T \text{ with eigenvalue } \lambda_1 = 2,$$

$$\mathbf{v}_2 = [1 \quad -1]^T \text{ with eigenvalue } \lambda_2 = -3.$$

So we have one positive and one negative eigenvalue. For the positive eigenvalue we get the solution

$$\mathbf{x}_1 = C_1 \begin{bmatrix} 4 \\ 1 \end{bmatrix} e^{\sqrt{2}t} + C_2 \begin{bmatrix} 4 \\ 1 \end{bmatrix} e^{-\sqrt{2}t}.$$

We could write the solution for the negative eigenvalue as

$$\mathbf{x}_2 = C_3 \begin{bmatrix} 1 \\ -1 \end{bmatrix} e^{i\sqrt{3}t} + C_4 \begin{bmatrix} 1 \\ -1 \end{bmatrix} e^{-i\sqrt{3}t},$$

but as discussed above, this is not so convenient because then  $C_3$  and  $C_4$  have to be complex. So instead we use the above result and write the solution in terms of sines and cosines:

$$\mathbf{x}_2 = C_3 \begin{bmatrix} 1 \\ -1 \end{bmatrix} \cos(\sqrt{3}t) + C_4 \begin{bmatrix} 1 \\ -1 \end{bmatrix} \sin(\sqrt{3}t).$$

Adding all the solutions ( $\mathbf{x}_1 + \mathbf{x}_2$ ), we get the general solution:

$$\begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} 4 \\ 1 \end{bmatrix} (C_1 e^{\sqrt{2}t} + C_2 e^{-\sqrt{2}t}) + \begin{bmatrix} 1 \\ -1 \end{bmatrix} (C_3 \cos(\sqrt{3}t) + C_4 \sin(\sqrt{3}t)),$$

where  $C_1$ ,  $C_2$ ,  $C_3$  and  $C_4$  are *real* constants.

The above ideas can be formalised in the following procedure, which also tells you what to do when an eigenvalue of the matrix of coefficients is zero.

#### Procedure 4 Solving a second-order homogeneous linear system

To solve a system  $\ddot{\mathbf{x}} = \mathbf{A}\mathbf{x}$ , where  $\mathbf{A}$  is an  $n \times n$  matrix with  $n$  distinct real eigenvalues, do the following.

1. Find the eigenvalues  $\lambda_1, \lambda_2, \dots, \lambda_n$  of  $\mathbf{A}$ , and a corresponding set of eigenvectors  $\mathbf{v}_1, \mathbf{v}_2, \dots, \mathbf{v}_n$ .
2. Each *positive* eigenvalue  $\lambda$ , corresponding to an eigenvector  $\mathbf{v}$ , gives rise to two linearly independent solutions

$$\mathbf{v}e^{\sqrt{\lambda}t} \quad \text{and} \quad \mathbf{v}e^{-\sqrt{\lambda}t}.$$

Each *negative* eigenvalue  $\lambda$ , corresponding to an eigenvector  $\mathbf{v}$ , gives rise to two linearly independent solutions

$$\mathbf{v} \cos(\sqrt{-\lambda}t) \quad \text{and} \quad \mathbf{v} \sin(\sqrt{-\lambda}t).$$

A *zero* eigenvalue corresponding to an eigenvector  $\mathbf{v}$  gives rise to two linearly independent solutions

$$\mathbf{v} \quad \text{and} \quad \mathbf{v}t.$$

3. The general solution is then an arbitrary linear combination of the  $2n$  linearly independent solutions found in Step 2, involving  $2n$  arbitrary real constants.

Complex eigenvalues and repeated real eigenvalues are not discussed here, but they can be dealt with by generalising what we have discussed.

We do not prove this here, but you can verify it in any particular case (see Example 10 below).

We illustrate this procedure in the following example.

### Example 10

Find the general solution of the system of differential equations

$$\begin{cases} \ddot{x} = 3x + 2y + 2z, \\ \ddot{y} = 2x + 2y, \\ \ddot{z} = 2x + 4z. \end{cases}$$

### Solution

The matrix of coefficients is

$$\mathbf{A} = \begin{bmatrix} 3 & 2 & 2 \\ 2 & 2 & 0 \\ 2 & 0 & 4 \end{bmatrix}.$$

The eigenvectors of  $\mathbf{A}$  can be shown to be

$$\begin{aligned} &[2 \ 1 \ 2]^T \text{ with eigenvalue } \lambda = 6, \\ &[1 \ 2 \ -2]^T \text{ with eigenvalue } \lambda = 3, \\ &[-2 \ 2 \ 1]^T \text{ with eigenvalue } \lambda = 0. \end{aligned}$$

It follows from Procedure 4 that the general solution of the system is

$$\begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 2 \\ 1 \\ 2 \end{bmatrix} (C_1 e^{\sqrt{6}t} + C_2 e^{-\sqrt{6}t}) + \begin{bmatrix} 1 \\ 2 \\ -2 \end{bmatrix} (C_3 e^{\sqrt{3}t} + C_4 e^{-\sqrt{3}t}) + \begin{bmatrix} -2 \\ 2 \\ 1 \end{bmatrix} (C_5 + C_6 t).$$

You may like to verify that  $[-2 \ 2 \ 1]^T$  and  $[-2 \ 2 \ 1]^T t$  are both solutions of the system.

### Exercise 14

Find the general solution of the system of differential equations

$$\begin{cases} \ddot{x} = 2x + y - z, \\ \ddot{y} = -3y + 2z, \\ \ddot{z} = 4z. \end{cases}$$

(Hint: The eigenvectors of  $\begin{bmatrix} 2 & 1 & -1 \\ 0 & -3 & 2 \\ 0 & 0 & 4 \end{bmatrix}$  are  $\begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}$ ,  $\begin{bmatrix} 1 \\ -5 \\ 0 \end{bmatrix}$  and  $\begin{bmatrix} -5 \\ 4 \\ 14 \end{bmatrix}$ , with corresponding eigenvalues 2,  $-3$  and 4.)

In the next example and exercise, we first find the general solution of a system of second-order differential equations, then find a particular solution for given initial conditions. This will be relevant for Section 3 on normal modes.

**Example 11**

(a) Find the general solution of the system of differential equations

$$\begin{cases} \ddot{x} = -3x + y, \\ \ddot{y} = x - 3y. \end{cases}$$

(Hint: Eigenvectors of  $\begin{bmatrix} -3 & 1 \\ 1 & -3 \end{bmatrix}$  can be shown to be

$$\mathbf{v}_1 = \begin{bmatrix} 1 & 1 \end{bmatrix}^T \text{ with eigenvalue } \lambda_2 = -2,$$

$$\mathbf{v}_2 = \begin{bmatrix} 1 & -1 \end{bmatrix}^T \text{ with eigenvalue } \lambda_1 = -4.)$$

(b) Find the particular solution that satisfies the initial conditions  $\mathbf{x}(0) = \mathbf{v}_1$ ,  $\dot{\mathbf{x}}(0) = \mathbf{0}$ , where  $\mathbf{v}_1$  is the eigenvector given in the hint.

**Solution**

(a) Using the hint, we see that there are two negative eigenvalues. The first eigenvalue gives the term

$$\mathbf{x}_1 = C_1 \begin{bmatrix} 1 \\ 1 \end{bmatrix} \cos(\sqrt{2}t) + C_2 \begin{bmatrix} 1 \\ 1 \end{bmatrix} \sin(\sqrt{2}t).$$

The second eigenvalue gives the term

$$\mathbf{x}_2 = C_3 \begin{bmatrix} 1 \\ -1 \end{bmatrix} \cos(2t) + C_4 \begin{bmatrix} 1 \\ -1 \end{bmatrix} \sin(2t).$$

Adding the solutions ( $\mathbf{x}_1 + \mathbf{x}_2$ ), we get the general solution:

$$\begin{aligned} \mathbf{x}(t) &= \begin{bmatrix} 1 \\ 1 \end{bmatrix} (C_1 \cos(\sqrt{2}t) + C_2 \sin(\sqrt{2}t)) \\ &\quad + \begin{bmatrix} 1 \\ -1 \end{bmatrix} (C_3 \cos(2t) + C_4 \sin(2t)). \end{aligned}$$

(b) Setting  $t = 0$  in the general solution gives

$$\mathbf{x}(0) = C_1 \begin{bmatrix} 1 \\ 1 \end{bmatrix} + C_3 \begin{bmatrix} 1 \\ -1 \end{bmatrix} = \begin{bmatrix} C_1 + C_3 \\ C_1 - C_3 \end{bmatrix}.$$

Using the initial condition  $\mathbf{x}(0) = \mathbf{v}_1$  then gives

$$\begin{bmatrix} 1 \\ 1 \end{bmatrix} = \begin{bmatrix} C_1 + C_3 \\ C_1 - C_3 \end{bmatrix}.$$

Hence  $C_1 + C_3 = 1$  and  $C_1 - C_3 = 1$ , which have solution  $C_3 = 0$ ,  $C_1 = 1$ .

Now, differentiating the general solution with respect to  $t$ , we get

$$\begin{aligned} \dot{\mathbf{x}}(t) &= \sqrt{2} \begin{bmatrix} 1 \\ 1 \end{bmatrix} (-C_1 \sin(\sqrt{2}t) + C_2 \cos(\sqrt{2}t)) \\ &\quad + 2 \begin{bmatrix} 1 \\ -1 \end{bmatrix} (-C_3 \sin(2t) + C_4 \cos(2t)), \end{aligned}$$

and then setting  $t = 0$  and using the initial condition  $\dot{\mathbf{x}}(0) = \mathbf{0}$  gives

$$\dot{\mathbf{x}}(0) = \mathbf{0} = \sqrt{2}C_2 \begin{bmatrix} 1 \\ 1 \end{bmatrix} + 2C_4 \begin{bmatrix} 1 \\ -1 \end{bmatrix}.$$

This gives  $\sqrt{2}C_2 + 2C_4 = 0$  and  $\sqrt{2}C_2 - 2C_4 = 0$ , which have solution  $C_2 = C_4 = 0$ .

So we have  $C_2 = C_3 = C_4 = 0$  and  $C_1 = 1$ . Substituting into the general solution gives

$$\mathbf{x}(t) = \begin{bmatrix} 1 \\ 1 \end{bmatrix} \cos(\sqrt{2}t).$$

Clearly this satisfies the initial conditions.

### Exercise 15

For the system in Example 11, find the particular solution that satisfies the initial conditions  $\mathbf{x}(0) = \mathbf{v}_2$ ,  $\dot{\mathbf{x}}(0) = \mathbf{0}$ , where  $\mathbf{v}_2$  is the eigenvector given in the hint.

The particular solutions found in Example 11(b) and Exercise 15 are called *normal mode* solutions and are the topic of the next section.

## 3 Normal modes

In Subsection 2.1 we described a compound harmonic oscillator consisting of two masses and two springs (see Figure 5). We mentioned that the equations of motion for this system have special solutions, called normal modes, in which the two masses oscillate at the same frequency. Here we show (in Subsection 3.1) how these normal mode solutions are obtained in a simple system, and how they are combined to describe the general solution of the equations of motion.

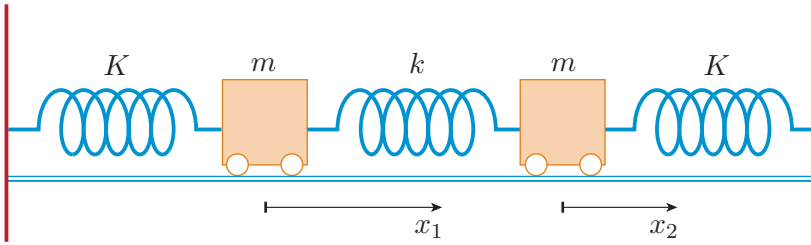
In Subsection 3.2 we consider an important scientific application of these ideas, by applying normal modes to describe the oscillations of certain types of simple molecule. These oscillations help scientists to explain how the Earth's atmosphere traps so much of the heat from the Sun, due to the *greenhouse effect*.

### 3.1 Motion of a simple two-mass system

#### An oscillating system and its equations of motion

We begin by describing the equations of motion for a rather special oscillating system.

Consider a system of two particles with equal masses  $m$  moving without friction on a rail, connected to three springs as illustrated in Figure 8. The two outer springs have spring stiffness  $K$  and are connected to rigid supports at either end. The middle spring has stiffness  $k$ . The displacements of the masses to the right of their equilibrium positions are  $x_1$  and  $x_2$  (for the left and right masses, respectively).



**Figure 8** Two masses slide without friction on a rail; the force on each mass is provided by two springs

The equations of motion for this system are derived from Newton's second law and can be shown to be

$$m \begin{bmatrix} \ddot{x}_1 \\ \ddot{x}_2 \end{bmatrix} = \begin{bmatrix} -(k+K) & k \\ k & -(k+K) \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix}. \quad (38)$$

We do not expect you to be able to derive such equations – they will be given to you if needed in assessment questions. However, the following is an argument that makes them plausible. (Don't worry if you can't follow this argument; it is not important for what follows).

From Newton's second law and equation (38), the force on the left-hand mass is seen to be

$$F_1 = -(k+K)x_1 + kx_2.$$

This makes sense because if we displace the left-hand mass to the right by a distance  $x_1$  and leave the right-hand mass fixed so  $x_2 = 0$ , then  $F_1 = -(k+K)x_1$ , so the right-hand mass is pushed in the opposite direction by both springs attached to it. Further, if we hold the left-hand mass fixed ( $x_1 = 0$ ) and displace the right-hand mass to the right by  $x_2$ , then  $F_1 = kx_2$ , so the left-hand mass is pulled to the right by the middle spring.

A similar line of reasoning can be used to show that it is plausible that the force on the right-hand mass is  $F_2 = kx_1 - (k+K)x_2$ .

Forces and displacements are normally written as vectors. But as the motion is in one direction here, we simply consider the components of the forces and displacements in that direction.

We now move on to solving these equations of motion.

### Solving the equations of motion

Equation (38) is a system of equations of the form  $\ddot{\mathbf{x}} = \mathbf{A}\mathbf{x}$ , where the matrix of coefficients is

$$\mathbf{A} = \frac{1}{m} \begin{bmatrix} -(k+K) & k \\ k & -(k+K) \end{bmatrix}.$$

From Section 2 we know that solutions are constructed using the eigenvalues and eigenvectors of  $\mathbf{A}$ .

You studied how to calculate eigenvectors and eigenvalues in Unit 5. The method is straightforward but a little tedious, so we will simply tell you what they are, and let you verify for yourself that they satisfy  $\mathbf{A}\mathbf{v} = \lambda\mathbf{v}$ .

The eigenvectors are

$$\mathbf{v}_1 = \begin{bmatrix} 1 \\ 1 \end{bmatrix}, \quad \mathbf{v}_2 = \begin{bmatrix} 1 \\ -1 \end{bmatrix}, \quad (39)$$

and the corresponding eigenvalues are

$$\lambda_1 = -\frac{K}{m}, \quad \lambda_2 = -\frac{K + 2k}{m}. \quad (40)$$

Note that the components of both eigenvectors  $\mathbf{v} = [v_1 \ v_2]$  satisfy  $|v_1| = |v_2|$ . This is because the system in Figure 8 has a reflection symmetry about its midpoint, i.e. the symmetry of the system is linked to the solution to the eigenvalue problem.

We note that because  $K$ ,  $k$  and  $m$  are all positive, both of the eigenvalues are negative. Hence from Procedure 4, the solutions of  $\ddot{\mathbf{x}} = \mathbf{A}\mathbf{x}$  are of the form

$$\mathbf{x}(t) = \cos(\omega_i t) \mathbf{v}_i, \quad \dot{\mathbf{x}}(t) = \sin(\omega_i t) \mathbf{v}_i,$$

where

$$\omega_i = \sqrt{-\lambda_i}. \quad (41)$$

The general solution is a linear combination of all of these solutions. We have four solutions, two for each eigenvalue:

$$\begin{aligned} \mathbf{x}(t) = & (C_1 \sin(\omega_1 t) + D_1 \cos(\omega_1 t)) \mathbf{v}_1 \\ & + (C_2 \sin(\omega_2 t) + D_2 \cos(\omega_2 t)) \mathbf{v}_2, \end{aligned} \quad (42)$$

where  $C_1$ ,  $D_1$ ,  $C_2$  and  $D_2$  are arbitrary real constants.

### Normal modes of vibration

We note that the general solution, equation (42), can be written as the sum of two terms

$$\mathbf{x}(t) = \mathbf{x}_{\omega_1}(t) + \mathbf{x}_{\omega_2}(t),$$

where

$$\begin{aligned} \mathbf{x}_{\omega_1}(t) &= (C_1 \sin(\omega_1 t) + D_1 \cos(\omega_1 t)) \mathbf{v}_1, \\ \mathbf{x}_{\omega_2}(t) &= (C_2 \sin(\omega_2 t) + D_2 \cos(\omega_2 t)) \mathbf{v}_2. \end{aligned}$$

$\mathbf{x}_{\omega_1}(t)$  is the part of the solution that oscillates with angular frequency  $\omega_1$ , and  $\mathbf{x}_{\omega_2}(t)$  is the part of the solution that oscillates with angular frequency  $\omega_2$ . These are called the **normal modes** of oscillation of the system.

If we were to set  $C_2 = D_2 = 0$  in the general solution, then  $\mathbf{x}(t) = \mathbf{x}_{\omega_1}(t)$ , and we would have a solution where both of the masses oscillated with the same angular frequency  $\omega_1$ . Likewise, if we set  $C_1 = D_1 = 0$ , then both masses would oscillate with angular frequency  $\omega_2$ . In fact, this defines what we mean by a normal mode for a system with any number of dependent variables.

### Normal mode

A *normal mode of oscillation* is one in which all of the coordinates of the system oscillate sinusoidally with the same angular frequency.

Suppose that we were to choose initial conditions so that  $\mathbf{x}(t) = \mathbf{x}_{\omega_1}(t)$ . Then since  $\mathbf{v}_1 = [1 \ 1]^T$ , we have

$$\begin{bmatrix} x_1(t) \\ x_2(t) \end{bmatrix} = \mathbf{x}_{\omega_1}(t) = (C_1 \sin(\omega_1 t) + D_1 \cos(\omega_1 t)) \begin{bmatrix} 1 \\ 1 \end{bmatrix},$$

hence the displacement of the masses would obey the equations

$$x_1(t) = x_2(t) = (C_1 \sin(\omega_1 t) + D_1 \cos(\omega_1 t)) \quad \text{for } \mathbf{x}(t) = \mathbf{x}_{\omega_1}(t). \quad (43)$$

There are two coefficients,  $C_1$  and  $D_1$ , that are determined by the initial conditions. We solved a system just like this, for the special case of  $m = 1$ ,  $k = 1$ ,  $K = 2$ , in Example 11. In part (b) we found that the initial conditions  $\mathbf{x}(0) = \mathbf{v}_1$  and  $\dot{\mathbf{x}}(0) = \mathbf{0}$  lead to the normal mode solution  $\mathbf{x}(t) = \cos(\omega_1 t) \mathbf{v}_1$ . In fact, these initial conditions give this normal mode solution for any values of  $m$ ,  $k$  and  $K$ .

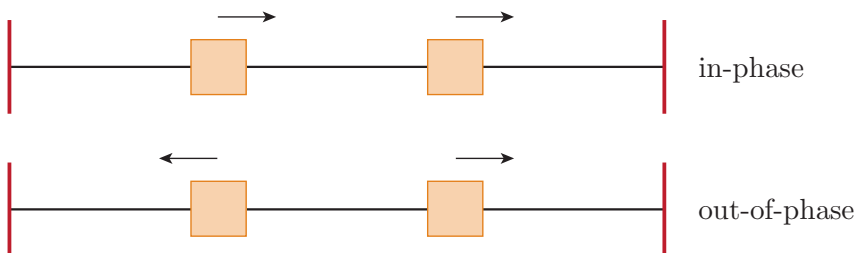
In fact,  $C_1$  and  $D_1$  determine the amplitude and phase of the oscillation.

Likewise, if we chose initial conditions so that  $\mathbf{x}(t) = \mathbf{x}_{\omega_2}(t)$ , then since  $\mathbf{v}_2 = [1 \ -1]^T$ , the displacements of the masses would obey the equations

$$x_1(t) = -x_2(t) = (C_2 \sin(\omega_2 t) + D_2 \cos(\omega_2 t)) \quad \text{for } \mathbf{x}(t) = \mathbf{x}_{\omega_2}(t). \quad (44)$$

Again, the two coefficients,  $C_2$  and  $D_2$  are determined by the initial conditions. For example, the initial conditions  $\mathbf{x}(0) = \mathbf{v}_2$  and  $\dot{\mathbf{x}}(0) = \mathbf{0}$  lead to the normal mode solution  $\mathbf{x}(t) = \cos(\omega_2 t) \mathbf{v}_2$ : see Exercise 15 for the special case of  $m = 1$ ,  $k = 1$ ,  $K = 2$ .

The normal mode with frequency  $\omega_1$  (equations (43)) is called the **symmetric** or **in-phase mode** of oscillation, because the masses always move in the same direction together. In this case  $x_1(t) = x_2(t)$ , hence the displacements of the masses are always equal. The normal mode with frequency  $\omega_2$  (equations (44)) is called the **antisymmetric** or **out-of-phase mode**, because the masses are always moving in opposite directions. In this case  $x_1(t) = -x_2(t)$ , hence the displacements are always of equal magnitude but opposite sign. The motion of the masses in the two normal modes is illustrated in Figure 9.



**Figure 9** Normal modes for the symmetric two-mass system

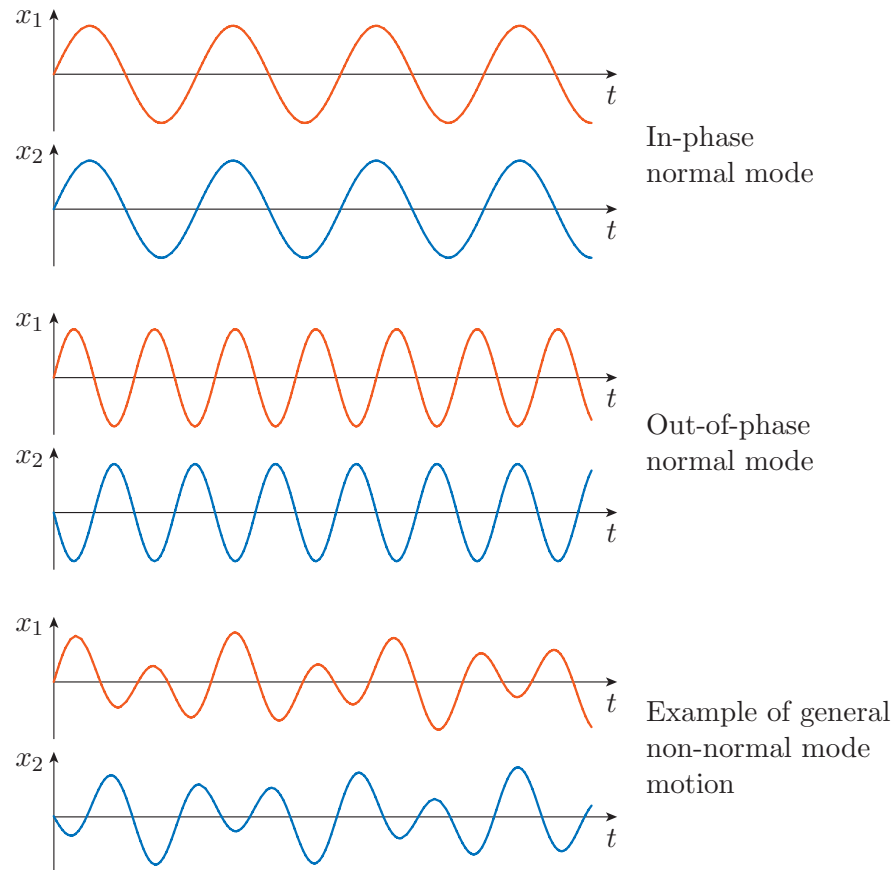
From equations (40) and (41), the angular frequencies of the normal modes are given by

$$\omega_1 = \sqrt{\frac{K}{m}} \text{ (in-phase)} \quad \text{and} \quad \omega_2 = \sqrt{\frac{K+2k}{m}} \text{ (out-of-phase)}. \quad (45)$$

Note that for the in-phase normal mode solution we have  $x_1(t) = x_2(t)$ , hence  $x_1(t) - x_2(t) = 0$ , so the length of the middle spring never changes. That is why the frequency of the in-phase normal mode,  $\omega_2$ , is the same as that for a simple harmonic oscillator with a spring stiffness equal to  $K$ .

Also note that since  $K$ ,  $k$  and  $m$  are all positive,  $\omega_2 > \omega_1$ , so the out-of-phase mode oscillates with a higher frequency than the in-phase mode. This is because each mass is being pulled by two springs instead of one as in the in-phase case.

The motion of each normal mode is a simple sinusoidal oscillation with a single angular frequency. Only special initial conditions like the ones given in Example 11 and Exercise 15 give rise to such normal mode motions. For arbitrary initial conditions (e.g.  $\mathbf{x}(0) = [1 \ 2]^T$ ,  $\dot{\mathbf{x}}(0) = [0 \ 1]^T$ ) the solution is a combination of sinusoidal oscillations with two different frequencies. In other words it is a linear combination of normal mode solutions, where the displacements of the two masses move in a seemingly complicated manner and are not proportional to each other.



**Figure 10** General non-normal mode motions are a combination of oscillations at the frequencies of the normal modes

Typical motions for these cases are illustrated in Figure 10. Note that these normal modes are rather special, since the in-phase mode has  $x_1(t) = x_2(t)$  and the out-of-phase mode has  $x_1(t) = -x_2(t)$ . This special motion arises because of the reflection symmetry of the system – the masses and spring constants on either side are equal. For  $m_1 \neq m_2$  or different springs on either side, there are still two normal modes, where both masses oscillate with the same frequency and  $x_1(t) = \alpha x_2(t)$ , for some number  $\alpha$ . The in-phase mode is characterised by  $\alpha$  being positive, and the out-of-phase mode is characterised by  $\alpha$  being negative. The angular frequency of the out-of-phase normal mode is always greater than that of the in-phase normal mode.

### Example 12

A system of two masses has its motion given by the system

$$\begin{bmatrix} \ddot{x}_1 \\ \ddot{x}_2 \end{bmatrix} = \begin{bmatrix} -2 & 2 \\ \frac{5}{2} & -6 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix}.$$

- (a) Write down the normal mode solutions and identify them as in-phase or out-of-phase.

(Hint:  $\mathbf{A} = \begin{bmatrix} -2 & 2 \\ \frac{5}{2} & -6 \end{bmatrix}$  has eigenvectors  $\mathbf{v}_1 = [2 \ 1]^T$  and  $\mathbf{v}_2 = [-\frac{2}{5} \ 1]^T$ , with corresponding eigenvalues  $\lambda_1 = -1$  and  $\lambda_2 = -7$ .)

- (b) Which motion will initial conditions  $\mathbf{x}(0) = \mathbf{v}_1$ ,  $\dot{\mathbf{x}}(0) = \mathbf{0}$  give rise to?

### Solution

- (a) The matrix of coefficients has negative eigenvalues. The first eigenvalue gives the term

$$\mathbf{x}_1 = \begin{bmatrix} 2 \\ 1 \end{bmatrix} (C_1 \cos(t) + C_2 \sin(t)),$$

where  $C_1$  and  $C_2$  are arbitrary real constants. This is the in-phase normal mode solution, because the components of  $\mathbf{v}_1$  have the same sign. The second eigenvalue gives the term

$$\mathbf{x}_2 = \begin{bmatrix} -\frac{2}{5} \\ 1 \end{bmatrix} (C_3 \cos(\sqrt{7}t) + C_4 \sin(\sqrt{7}t)),$$

where  $C_3$  and  $C_4$  are arbitrary real constants. This is the out-of-phase normal mode solution, because the components of  $\mathbf{v}_2$  have opposite signs.

We note that the angular frequency of the out-of-phase normal mode ( $\omega_{\text{out}} = \sqrt{7}$ ) is greater than the in-phase normal mode ( $\omega_{\text{in}} = 1$ ), as expected.

- (b) Following the same reasoning as in Example 11 and Exercise 15, these initial conditions give rise to the normal mode solution

$$\mathbf{x}(t) = \mathbf{v}_1 \cos(t) = \begin{bmatrix} 2 \\ 1 \end{bmatrix} \cos(t).$$

It is obvious that this solution satisfies the given initial conditions.

Note that this does not correspond to a *symmetric* two-mass system connected by springs.

**Exercise 16**

A system of two masses connected by springs has two normal mode angular frequencies:  $\omega_1 = 2$  and  $\omega_2 = 4$ . Which is the in-phase frequency?

**Exercise 17**

A system of two masses connected by springs has a characteristic matrix with eigenvectors  $\mathbf{v}_1 = [1 \ -2]^T$  and  $\mathbf{v}_2 = [1 \ 4]^T$ . Which gives rise to the in-phase mode?

**Exercise 18**

Two masses connected by springs have their motion given by the system

$$\begin{bmatrix} \ddot{x}_1 \\ \ddot{x}_2 \end{bmatrix} = \begin{bmatrix} -4 & 1 \\ 1 & -4 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix}.$$

- (a) Write down the normal mode solutions and identify them as in-phase or out-of-phase.

(Hint:  $\mathbf{A} = \begin{bmatrix} -4 & 1 \\ 1 & -4 \end{bmatrix}$  has eigenvectors  $\mathbf{v}_1 = [1 \ 1]^T$  and  $\mathbf{v}_2 = [1 \ -1]^T$ , with corresponding eigenvalues  $\lambda_1 = -3$  and  $\lambda_2 = -5$ .)

- (b) Which motion will the initial conditions  $\mathbf{x}(0) = \mathbf{v}_1$ ,  $\dot{\mathbf{x}}(0) = \mathbf{0}$  give rise to?

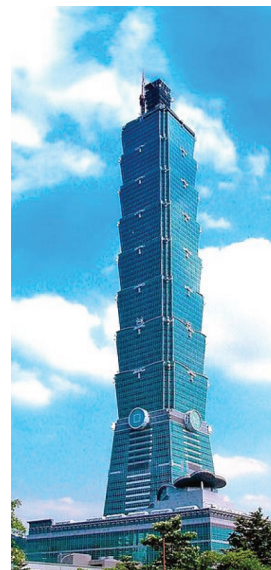
**Normal modes in engineering**

You may have seen footage of a wine glass breaking when someone sings or plays a loud (usually amplified) note at exactly the right pitch. This occurs because the frequency of the note matches a ‘natural frequency’ of the wine glass, causing it to vibrate. Other objects – from molecules to buildings – have ‘natural frequencies’ at which they prefer to vibrate. These ‘natural frequencies’ are in fact the normal mode frequencies of the objects.

The vibration of many objects can be modelled by systems of differential equations that are generalisations of the type that we have considered here. The general solution is a linear combination of normal mode solutions, each of which is a sinusoidal motion with a single frequency – the normal mode frequency.

It is very important for engineers to be able to predict the normal modes of vibration of the objects that they build, because the normal mode frequencies of a structure are those frequencies at which the structure will tend to resonate (i.e. vibrate with large amplitude).

In structural engineering, it is imperative that a building's normal mode frequencies do not match the frequencies of expected earthquakes, otherwise an earthquake may make the structure vibrate, causing damage. The periodic variation of wind gusts can be another cause of resonant vibration in structures like bridges or very tall buildings, and measures have to be taken to avoid the vibrations becoming too large. For example, the 509 m tall Taipei 101 building in Taiwan (the tallest in the world until 2010), shown in Figure 11, has a 660 tonne steel pendulum suspended from its 92nd floor to dampen resonant vibrations caused by wind gusts.

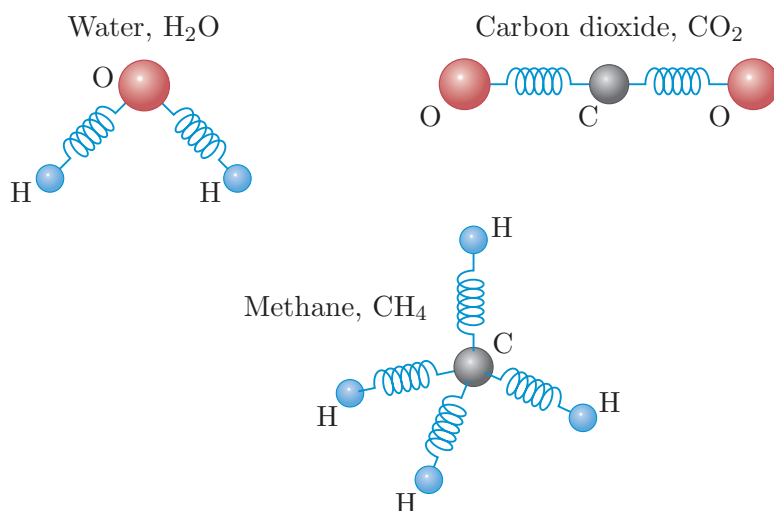


**Figure 11** The Taipei 101 building

### 3.2 An application: vibrations of simple molecules

This optional subsection discusses one of the most important physical applications of normal modes. **You will not be assessed on any of the material in this subsection.**

Simple molecules consist of a small number of atoms held together by 'chemical bonds'. Figure 12 illustrates some simple molecules. You can think of the atoms as behaving like point masses. The chemical bonds are not entirely rigid: they can behave like springs, so that the molecule can oscillate. We end this unit by considering a model for the frequency of oscillation of some simple molecules.



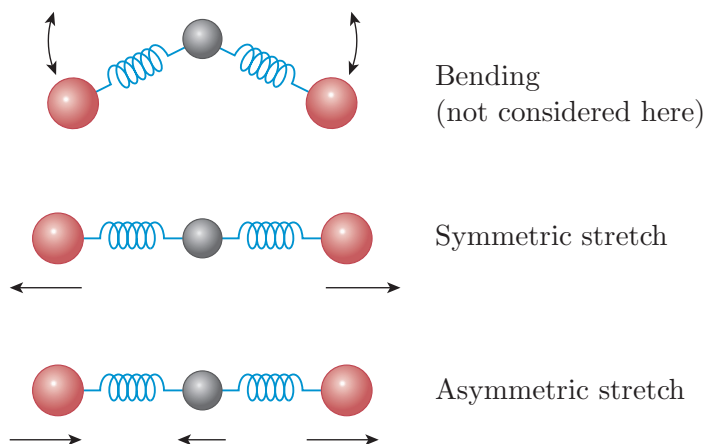
**Figure 12** Some simple molecules, modelled as masses connected by springs

### Molecular vibrations and the greenhouse effect

Oscillations of molecules occur at very high frequencies, typically of the order of  $10^{13}$  oscillations per second. Such rapid oscillations cannot be recorded by any mechanical instrument, but they do correspond to the frequency at which electric fields oscillate in infrared light (a type of light that is not visible to the human eye). In fact, molecules absorb infrared light by a resonance effect.

This absorption in the infrared is important for determining the climate of the Earth. Light from the Sun passes through the atmosphere and warms the surface of the Earth. The warm surface of the Earth radiates infrared light, which cannot escape into space because it is absorbed by molecules in the atmosphere (especially water and carbon dioxide). If it were not for this absorption, the Earth would be considerably colder. This is called the greenhouse effect. It is topical because of concerns that too much carbon dioxide in the atmosphere may cause a dangerous increase in the greenhouse effect.

The frequencies of oscillation of molecules depend on their geometry and on the masses of the atoms and the ‘spring stiffnesses’ of the chemical bonds. For most molecules, the normal modes of vibration form a complicated three-dimensional motion. In this subsection we consider only the vibrations of a simple type of molecule, called a *linear triatomic molecule*. Carbon dioxide is an example of such a molecule. Its small oscillations can be of two types. There are bending vibrations, and there are motions that involve stretching of the chemical bonds while the atoms move along the same line (see Figure 13). In this subsection we confine our attention to the stretching motion, and we do not consider the bending modes of vibration.



**Figure 13** Modes of vibration of carbon dioxide

Our model for the carbon dioxide molecule consists of three atoms in a row, with masses  $m$ ,  $M$  and  $m$ . Each of the outer masses is connected to the central mass by a ‘spring’ with stiffness constant  $k$ . The displacements of the three masses from their equilibrium positions are  $x_1$ ,  $x_2$  and  $x_3$ , respectively, as shown in Figure 14. The forces due to the ‘springs’ can be shown to be  $F_1 = k(x_2 - x_1)$ ,  $F_2 = k(x_1 - x_2) + k(x_3 - x_2)$  and  $F_3 = k(x_2 - x_3)$ , so that the equations of motion are

$$m\ddot{x}_1 = k(x_2 - x_1),$$

$$M\ddot{x}_2 = k(x_1 - 2x_2 + x_3),$$

$$m\ddot{x}_3 = k(x_2 - x_3).$$

This system is equivalent to the matrix equation

$$\ddot{\mathbf{x}} = \mathbf{A}\mathbf{x}, \quad \text{where } \mathbf{A} = \begin{bmatrix} -\frac{k}{m} & \frac{k}{m} & 0 \\ \frac{k}{M} & -\frac{2k}{M} & \frac{k}{M} \\ 0 & \frac{k}{m} & -\frac{k}{m} \end{bmatrix}.$$

Once again the general solution and normal modes are determined from the eigenvalues and eigenvectors of the coefficient matrix  $\mathbf{A}$ . It is not difficult to show that the characteristic equation is

$$\det(\mathbf{A} - \lambda\mathbf{I}) = \lambda \left( \lambda + \frac{k}{m} \right) \left( \lambda + \frac{k}{m} + 2\frac{k}{M} \right) = 0.$$

From this we deduce that the eigenvalues are

$$\lambda_1 = -\frac{k}{m}, \quad \lambda_2 = -\frac{k}{m} \left( 1 + \frac{2m}{M} \right), \quad \lambda_3 = 0.$$

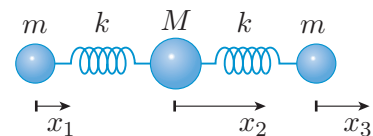
$\lambda_1$  and  $\lambda_2$  are both negative, so they correspond to oscillations or vibrations of the molecule with frequencies

$$\omega_1 = \sqrt{\frac{k}{m}} \quad \text{and} \quad \omega_2 = \sqrt{\frac{k}{m}} \sqrt{\frac{M + 2m}{M}}.$$

You are no doubt wondering what sort of motion the zero eigenvalue  $\lambda_3$  corresponds to. It turns out that its eigenvector is  $\mathbf{v}_3 = [1 \ 1 \ 1]^T$ . This has a simple physical interpretation: it corresponds to a motion where all of the atoms move together in the same direction by the same amount, so the motion does not stretch either of the springs.

We now return to consider the modes of vibration. The vibration at frequency  $\omega_1$  corresponds to the eigenvector  $[1 \ 0 \ -1]^T$ , for which the motion is a *symmetric stretch*. The frequency  $\omega_2$  has eigenvector  $[1 \ -2m/M \ 1]^T$ , and corresponds to an *asymmetric stretch* normal mode (see Figure 13).

Consider how this works for carbon dioxide. The masses of atoms are measured in units of the mass of a hydrogen atom,  $m_H$ . Carbon dioxide (chemical symbol  $\text{CO}_2$ ) is a symmetric linear triatomic molecule, with two oxygen atoms, each of mass  $m = 16m_H$ , and a carbon atom, of mass  $M = 12m_H$ .



**Figure 14** Displacement of atoms from their equilibrium positions

As in earlier models, these equations are actually the components of the vector equations for forces and displacements, in the direction of motion.

The ratio of the two stretching vibrational frequencies is therefore expected to be

$$\frac{\omega_2}{\omega_1} = \sqrt{\frac{M+2m}{M}} = \sqrt{\frac{44}{12}} \simeq 1.9.$$

The frequencies can be investigated by looking for resonances in the absorption of carbon dioxide as the frequency of a light source is varied. In a unit favoured by spectroscopists ( $\text{cm}^{-1}$ ), the frequencies of vibration of carbon dioxide molecules are found to be 667, 1388 and 2349. The smallest of these is the frequency of bending vibrations, and the other two correspond to the two stretching vibrations. The ratios of the observed stretching vibration frequencies is  $2349/1388 \simeq 1.7$ . This is quite close to 1.9, indicating that the simple mass and spring model is a reasonable model for the vibrations of the  $\text{CO}_2$  molecule.

To convert units of  $\text{cm}^{-1}$  to Hertz, multiply by the speed of light,  $c \simeq 3 \times 10^{10} \text{ cm s}^{-1}$ .

For a better model we need to turn to quantum mechanics and solve the Schrödinger equation for carbon dioxide.

## Learning outcomes

After studying this unit, you should be able to do the following.

- Understand and use the terminology associated with systems of linear constant-coefficient differential equations.
- Obtain the general solution of a homogeneous system of two or three first-order differential equations, by applying knowledge of the eigenvalues and eigenvectors of the coefficient matrix.
- Obtain a particular integral of an inhomogeneous system of two first-order differential equations in certain simple cases, by using a trial solution.
- Obtain the general solution of an inhomogeneous system of two or three first-order differential equations, by combining its complementary function and a particular integral.
- Apply given initial conditions to obtain the solution of an initial-value problem that features a system of two or three first-order differential equations.
- Obtain the general solution of a homogeneous system of two or three second-order equations, by applying knowledge of the eigenvalues and eigenvectors of the coefficient matrix.
- Obtain a particular solution of a pair of second-order homogeneous equations, by applying initial conditions.
- Identify normal mode solutions for systems of two masses.

# Solutions to exercises

## Solution to Exercise 1

$$(a) \begin{bmatrix} \dot{x} \\ \dot{y} \end{bmatrix} = \begin{bmatrix} 2 & 1 \\ 1 & 0 \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix} + \begin{bmatrix} 1 \\ -2 \end{bmatrix}; \text{ inhomogeneous.}$$

$$(b) \begin{bmatrix} \dot{x} \\ \dot{y} \end{bmatrix} = \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix} + \begin{bmatrix} 0 \\ t \end{bmatrix}; \text{ inhomogeneous.}$$

$$(c) \begin{bmatrix} \dot{x} \\ \dot{y} \\ \dot{z} \end{bmatrix} = \begin{bmatrix} 5 & 0 & 0 \\ 1 & 2 & 1 \\ 1 & 1 & 2 \end{bmatrix} \begin{bmatrix} x \\ y \\ z \end{bmatrix}; \text{ homogeneous.}$$

## Solution to Exercise 2

(a) The matrix of coefficients is

$$\mathbf{A} = \begin{bmatrix} 3 & 2 \\ 1 & 4 \end{bmatrix}.$$

Using the given eigenvalues and eigenvectors, we can construct two independent solutions  $\mathbf{v}e^{\lambda t}$ :

$$\mathbf{x}_1 = \begin{bmatrix} 1 \\ 1 \end{bmatrix} e^{5t} \quad \text{and} \quad \mathbf{x}_2 = \begin{bmatrix} -2 \\ 1 \end{bmatrix} e^{2t}.$$

(b) The general solution is a general linear combination of the two independent solutions:

$$\begin{bmatrix} x \\ y \end{bmatrix} = \alpha \begin{bmatrix} 1 \\ 1 \end{bmatrix} e^{5t} + \beta \begin{bmatrix} -2 \\ 1 \end{bmatrix} e^{2t},$$

where  $\alpha$  and  $\beta$  are arbitrary constants.

## Solution to Exercise 3

The matrix of coefficients is

$$\mathbf{A} = \begin{bmatrix} 5 & 2 \\ 2 & 5 \end{bmatrix}.$$

We are given that the eigenvectors of  $\mathbf{A}$  are  $[1 \ 1]^T$  with corresponding eigenvalue  $\lambda = 7$ , and  $[1 \ -1]^T$  with corresponding eigenvalue  $\lambda = 3$ . The general solution is therefore

$$\begin{bmatrix} x \\ y \end{bmatrix} = \alpha \begin{bmatrix} 1 \\ 1 \end{bmatrix} e^{7t} + \beta \begin{bmatrix} 1 \\ -1 \end{bmatrix} e^{3t},$$

where  $\alpha$  and  $\beta$  are arbitrary constants.

**Solution to Exercise 4**

- (a) In Exercise 3 we showed that the general solution of these differential equations is

$$\begin{bmatrix} x \\ y \end{bmatrix} = \alpha \begin{bmatrix} 1 \\ 1 \end{bmatrix} e^{7t} + \beta \begin{bmatrix} 1 \\ -1 \end{bmatrix} e^{3t}.$$

Since  $x = 4$  and  $y = 0$  when  $t = 0$ , we have

$$\begin{cases} 4 = \alpha + \beta, \\ 0 = \alpha - \beta. \end{cases}$$

Thus  $\alpha = 2$  and  $\beta = 2$ , so

$$\begin{aligned} \begin{bmatrix} x \\ y \end{bmatrix} &= 2 \begin{bmatrix} 1 \\ 1 \end{bmatrix} e^{7t} + 2 \begin{bmatrix} 1 \\ -1 \end{bmatrix} e^{3t} \\ &= \begin{bmatrix} 2 \\ 2 \end{bmatrix} e^{7t} + \begin{bmatrix} 2 \\ -2 \end{bmatrix} e^{3t}. \end{aligned}$$

- (b) For large  $t$ ,  $e^{7t}$  is much greater than  $e^{3t}$ , hence  $x \simeq 2e^{7t}$  and  $y \simeq 2e^{7t}$ . Hence the solution approaches the line  $x = y$  for large  $t$ .

**Solution to Exercise 5**

The matrix of coefficients is

$$\mathbf{A} = \begin{bmatrix} 5 & 0 & 0 \\ 1 & 2 & 1 \\ 1 & 1 & 2 \end{bmatrix}.$$

We are given that the eigenvectors of  $\mathbf{A}$  are  $[2 \ 1 \ 1]^T$ ,  $[0 \ 1 \ 1]^T$  and  $[0 \ 1 \ -1]^T$ , corresponding to the eigenvalues 5, 3 and 1.

The general solution is therefore

$$\begin{bmatrix} x \\ y \\ z \end{bmatrix} = \alpha \begin{bmatrix} 2 \\ 1 \\ 1 \end{bmatrix} e^{5t} + \beta \begin{bmatrix} 0 \\ 1 \\ 1 \end{bmatrix} e^{3t} + \gamma \begin{bmatrix} 0 \\ 1 \\ -1 \end{bmatrix} e^t.$$

Since  $x = 4$ ,  $y = 6$  and  $z = 0$  when  $t = 0$ , we have

$$\begin{cases} 4 = 2\alpha, \\ 6 = \alpha + \beta + \gamma, \\ 0 = \alpha + \beta - \gamma. \end{cases}$$

From this we can deduce that  $\alpha = 2$ ,  $\beta = 1$  and  $\gamma = 3$ , so

$$\begin{aligned} \begin{bmatrix} x \\ y \\ z \end{bmatrix} &= 2 \begin{bmatrix} 2 \\ 1 \\ 1 \end{bmatrix} e^{5t} + \begin{bmatrix} 0 \\ 1 \\ 1 \end{bmatrix} e^{3t} + 3 \begin{bmatrix} 0 \\ 1 \\ -1 \end{bmatrix} e^t \\ &= \begin{bmatrix} 4 \\ 2 \\ 2 \end{bmatrix} e^{5t} + \begin{bmatrix} 0 \\ 1 \\ 1 \end{bmatrix} e^{3t} + \begin{bmatrix} 0 \\ 3 \\ -3 \end{bmatrix} e^t. \end{aligned}$$

**Solution to Exercise 6**

(a) If  $z = 4 + 3i$ , then  $\bar{z} = 4 - 3i$ , which gives  $\bar{\bar{z}} = 4 + 3i$ .

Hence  $\bar{\bar{z}} = z$ .

(b) If  $z = 4 + 3i$ , then  $\operatorname{Re} z = 4$ . On the other hand,  $\bar{z} = 4 - 3i$ , which gives  $\frac{1}{2}(z + \bar{z}) = \frac{1}{2}((4 + 3i) + (4 - 3i)) = 4$ .

Hence  $\operatorname{Re} z = \frac{1}{2}(z + \bar{z})$ .

(c) If  $z = 4 + 3i$ , then  $\operatorname{Im} z = 3$ . On the other hand,  $\frac{1}{2i}(z - \bar{z}) = \frac{1}{2i}((4 + 3i) - (4 - 3i)) = \frac{6i}{2i} = 3$ .

Hence  $\operatorname{Im} z = \frac{1}{2i}(z - \bar{z})$ .

**Solution to Exercise 7**

(a) The matrix of coefficients is

$$\mathbf{A} = \begin{bmatrix} -3 & -2 \\ 4 & 1 \end{bmatrix}.$$

Using the given eigenvalues and eigenvectors, we obtain the general solution

$$\begin{bmatrix} x \\ y \end{bmatrix} = C \begin{bmatrix} 1 \\ -1 - i \end{bmatrix} e^{(-1+2i)t} + D \begin{bmatrix} 1 \\ -1 + i \end{bmatrix} e^{(-1-2i)t}.$$

Now

$$\begin{aligned} \begin{bmatrix} 1 \\ -1 - i \end{bmatrix} e^{(-1+2i)t} &= \begin{bmatrix} e^{-t}(\cos 2t + i \sin 2t) \\ (-1 - i)e^{-t}(\cos 2t + i \sin 2t) \end{bmatrix} \\ &= \begin{bmatrix} e^{-t} \cos 2t + ie^{-t} \sin 2t \\ e^{-t}(\sin 2t - \cos 2t) - ie^{-t}(\sin 2t + \cos 2t) \end{bmatrix}. \end{aligned}$$

So we have

$$\operatorname{Re} \left( \begin{bmatrix} 1 \\ -1 - i \end{bmatrix} e^{(-1+2i)t} \right) = \begin{bmatrix} \cos 2t \\ \sin 2t - \cos 2t \end{bmatrix} e^{-t},$$

$$\operatorname{Im} \left( \begin{bmatrix} 1 \\ -1 - i \end{bmatrix} e^{(-1+2i)t} \right) = \begin{bmatrix} \sin 2t \\ -\sin 2t - \cos 2t \end{bmatrix} e^{-t},$$

and the general real-valued solution can be written as

$$\begin{bmatrix} x \\ y \end{bmatrix} = \alpha \begin{bmatrix} \cos 2t \\ \sin 2t - \cos 2t \end{bmatrix} e^{-t} + \beta \begin{bmatrix} \sin 2t \\ -\sin 2t - \cos 2t \end{bmatrix} e^{-t}.$$

(b) Since  $x = y = 1$  when  $t = 0$ , we have

$$1 = \alpha \quad \text{and} \quad 1 = -\alpha - \beta,$$

so  $\alpha = 1$ ,  $\beta = -2$ , and the required particular solution is

$$\begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} \cos 2t - 2 \sin 2t \\ 3 \sin 2t + \cos 2t \end{bmatrix} e^{-t}.$$

**Solution to Exercise 8**

(a) Using the given eigenvalues and eigenvectors in Procedure 2, we have

$$\begin{aligned}
 \mathbf{v}e^{\lambda t} &= \begin{bmatrix} 1 \\ -i \\ i \end{bmatrix} e^{(1+i)t} = e^t \begin{bmatrix} 1 \\ -i \\ i \end{bmatrix} (\cos t + i \sin t) \\
 &= e^t \begin{bmatrix} \cos t + i \sin t \\ \sin t - i \cos t \\ -\sin t + i \cos t \end{bmatrix} \\
 &= e^t \underbrace{\begin{bmatrix} \cos t \\ \sin t \\ -\sin t \end{bmatrix}}_{\text{real part}} + i e^t \underbrace{\begin{bmatrix} \sin t \\ -\cos t \\ \cos t \end{bmatrix}}_{\text{imaginary part}}.
 \end{aligned}$$

Thus the general real-valued solution is

$$\begin{bmatrix} x \\ y \\ z \end{bmatrix} = C_1 \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix} e^{2t} + C_2 e^t \begin{bmatrix} \cos t \\ \sin t \\ -\sin t \end{bmatrix} + C_3 e^t \begin{bmatrix} \sin t \\ -\cos t \\ \cos t \end{bmatrix},$$

where  $C_1$ ,  $C_2$  and  $C_3$  are real constants.

(b) Putting  $x = y = 1$  and  $z = 2$  when  $t = 0$ , we have

$$\begin{bmatrix} 1 \\ 1 \\ 2 \end{bmatrix} = C_1 \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix} + C_2 \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix} + C_3 \begin{bmatrix} 0 \\ -1 \\ 1 \end{bmatrix},$$

so  $C_2 = 1$ ,  $C_3 = 2$  and  $C_1 - C_3 = 1$ , giving  $C_1 = 3$ . Thus the required solution is

$$\begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 0 \\ 3 \\ 0 \end{bmatrix} e^{2t} + \begin{bmatrix} \cos t + 2 \sin t \\ -2 \cos t + \sin t \\ 2 \cos t - \sin t \end{bmatrix} e^t.$$

**Solution to Exercise 9**

$$\begin{bmatrix} x \\ y \end{bmatrix} = \alpha \begin{bmatrix} 1 \\ 1 \end{bmatrix} e^{5t} + \beta \begin{bmatrix} -2 \\ 1 \end{bmatrix} e^{2t} + \begin{bmatrix} t + \frac{4}{5} \\ -2t - \frac{7}{10} \end{bmatrix}.$$

**Solution to Exercise 10**

From Example 1, the complementary function is

$$\begin{bmatrix} x_c \\ y_c \end{bmatrix} = \alpha \begin{bmatrix} 4 \\ 1 \end{bmatrix} e^{2t} + \beta \begin{bmatrix} 1 \\ -1 \end{bmatrix} e^{-3t}.$$

For a particular integral, we try

$$\begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} at + b \\ ct + d \end{bmatrix},$$

where  $a$ ,  $b$ ,  $c$ ,  $d$  are constants to be determined.

Substituting  $x = at + b$ ,  $y = ct + d$  into the differential equations gives

$$\begin{cases} a = (at + b) + 4(ct + d) - t + 2, \\ c = (at + b) - 2(ct + d) + 5t, \end{cases}$$

which become

$$\begin{cases} (a + 4c - 1)t + (b + 4d + 2 - a) = 0, \\ (a - 2c + 5)t + (b - 2d - c) = 0. \end{cases}$$

Equating the coefficients of  $t$  to zero gives

$$\begin{cases} a + 4c - 1 = 0, \\ a - 2c + 5 = 0, \end{cases}$$

which have the solution  $a = -3$ ,  $c = 1$ .

Equating the constant terms to zero, and putting  $a = -3$ ,  $c = 1$ , gives

$$\begin{cases} b + 4d + 5 = 0, \\ b - 2d - 1 = 0, \end{cases}$$

which have the solution  $b = -1$ ,  $d = -1$ .

Thus the required particular integral is

$$\begin{bmatrix} x_p \\ y_p \end{bmatrix} = \begin{bmatrix} -3t - 1 \\ t - 1 \end{bmatrix},$$

and the general solution is

$$\begin{aligned} \begin{bmatrix} x \\ y \end{bmatrix} &= \begin{bmatrix} x_c \\ y_c \end{bmatrix} + \begin{bmatrix} x_p \\ y_p \end{bmatrix} \\ &= \alpha \begin{bmatrix} 4 \\ 1 \end{bmatrix} e^{2t} + \beta \begin{bmatrix} 1 \\ -1 \end{bmatrix} e^{-3t} + \begin{bmatrix} -3t - 1 \\ t - 1 \end{bmatrix}. \end{aligned}$$

### Solution to Exercise 11

The complementary function is

$$\begin{bmatrix} x_c \\ y_c \end{bmatrix} = \alpha \begin{bmatrix} 4 \\ 1 \end{bmatrix} e^{2t} + \beta \begin{bmatrix} 1 \\ -1 \end{bmatrix} e^{-3t}.$$

For a particular integral, we try

$$\begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} a \\ b \end{bmatrix} e^{-t},$$

where  $a$  and  $b$  are constants to be determined.

Substituting  $x = ae^{-t}$ ,  $y = be^{-t}$  into the differential equations gives

$$\begin{cases} -ae^{-t} = ae^{-t} + 4be^{-t} + 4e^{-t}, \\ -be^{-t} = ae^{-t} - 2be^{-t} + 5e^{-t}, \end{cases}$$

which, on dividing by  $e^{-t}$  and rearranging, become

$$\begin{cases} 2a + 4b = -4, \\ a - b = -5. \end{cases}$$

These equations have the solution  $a = -4$ ,  $b = 1$ .

Thus the required particular integral is

$$\begin{bmatrix} x_p \\ y_p \end{bmatrix} = \begin{bmatrix} -4 \\ 1 \end{bmatrix} e^{-t},$$

and the general solution is

$$\begin{aligned} \begin{bmatrix} x \\ y \end{bmatrix} &= \begin{bmatrix} x_c \\ y_c \end{bmatrix} + \begin{bmatrix} x_p \\ y_p \end{bmatrix} \\ &= \alpha \begin{bmatrix} 4 \\ 1 \end{bmatrix} e^{2t} + \beta \begin{bmatrix} 1 \\ -1 \end{bmatrix} e^{-3t} + \begin{bmatrix} -4 \\ 1 \end{bmatrix} e^{-t}. \end{aligned}$$

### Solution to Exercise 12

(a) The matrix of coefficients is

$$\mathbf{A} = \begin{bmatrix} 2 & 3 \\ 2 & 1 \end{bmatrix}.$$

First, we find its eigenvalues. The characteristic equation is

$$\det(\mathbf{A} - \lambda \mathbf{I}) = \begin{vmatrix} 2 - \lambda & 3 \\ 2 & 1 - \lambda \end{vmatrix} = 0.$$

Expanding this gives  $(2 - \lambda)(1 - \lambda) - 6 = 0$ , which simplifies to  $\lambda^2 - 3\lambda - 4 = 0$ . This factorises as  $(\lambda - 4)(\lambda + 1) = 0$ , so the eigenvalues are  $\lambda = -1$  and  $\lambda = 4$ .

Now we find the eigenvectors.

- For  $\lambda = -1$ , let the eigenvector be  $\mathbf{v} = [x \ y]^T$ . Then the eigenvector equations are

$$(\mathbf{A} - \lambda \mathbf{I})\mathbf{v} = (\mathbf{A} + \mathbf{I})\mathbf{v} = \begin{bmatrix} 3 & 3 \\ 2 & 2 \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix} = \mathbf{0}.$$

These give the simultaneous equations

$$3x + 3y = 0 \quad \text{and} \quad 2x + 2y = 0,$$

which reduce to the single equation  $y = -x$ . So (setting  $x = 1$ ), an eigenvector corresponding to  $\lambda = -1$  is  $[1 \ -1]^T$ .

- For  $\lambda = 4$ , the eigenvector equations are

$$(\mathbf{A} - \lambda \mathbf{I})\mathbf{v} = (\mathbf{A} - 4\mathbf{I})\mathbf{v} = \begin{bmatrix} -2 & 3 \\ 2 & -3 \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix} = \mathbf{0},$$

which give

$$-2x + 3y = 0 \quad \text{and} \quad 2x - 3y = 0,$$

which reduce to the single equation  $y = 2x/3$ . So (setting  $x = 3$ ) an eigenvector corresponding to  $\lambda = 4$  is  $[3 \ 2]^T$ .

- (b) Using the calculated eigenvalues and eigenvectors, the complementary function is

$$\begin{bmatrix} x_c \\ y_c \end{bmatrix} = \alpha \begin{bmatrix} 1 \\ -1 \end{bmatrix} e^{-t} + \beta \begin{bmatrix} 3 \\ 2 \end{bmatrix} e^{4t}.$$

We try a particular integral

$$\begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} a \\ b \end{bmatrix} e^{2t}.$$

Then

$$\begin{cases} 2ae^{2t} = 2ae^{2t} + 3be^{2t} + e^{2t}, \\ 2be^{2t} = 2ae^{2t} + be^{2t} + 4e^{2t}, \end{cases}$$

which give  $3b + 1 = 0$  and  $b - 2a = 4$ , so  $b = -\frac{1}{3}$  and  $a = -\frac{13}{6}$ . The general solution is therefore

$$\begin{bmatrix} x \\ y \end{bmatrix} = \alpha \begin{bmatrix} 1 \\ -1 \end{bmatrix} e^{-t} + \beta \begin{bmatrix} 3 \\ 2 \end{bmatrix} e^{4t} - \frac{1}{6} \begin{bmatrix} 13 \\ 2 \end{bmatrix} e^{2t}.$$

Putting  $t = 0$ , we obtain

$$\frac{5}{6} = \alpha + 3\beta - \frac{13}{6}, \quad \frac{2}{3} = -\alpha + 2\beta - \frac{1}{3},$$

so  $\alpha + 3\beta = 3$  and  $-\alpha + 2\beta = 1$ , which give  $\alpha = \frac{3}{5}$  and  $\beta = \frac{4}{5}$ .

The required solution is therefore

$$\begin{bmatrix} x \\ y \end{bmatrix} = \frac{3}{5} \begin{bmatrix} 1 \\ -1 \end{bmatrix} e^{-t} + \frac{4}{5} \begin{bmatrix} 3 \\ 2 \end{bmatrix} e^{4t} - \frac{1}{6} \begin{bmatrix} 13 \\ 2 \end{bmatrix} e^{2t}.$$

- (c) As  $t$  gets large, the  $e^{4t}$  term will become larger than any of the other terms. Hence

$$\begin{bmatrix} x \\ y \end{bmatrix} \simeq \frac{4}{5} \begin{bmatrix} 3 \\ 2 \end{bmatrix} e^{4t} \quad \text{for large } t.$$

So  $x = \frac{12}{5}e^{4t}$  and  $y = \frac{8}{5}e^{4t}$ . Hence  $x/y = \frac{12}{8} = \frac{3}{2}$ , thus the solution will approach the line  $x = 3y/2$  for large  $t$ .

### Solution to Exercise 13

The matrix of coefficients is

$$\mathbf{A} = \begin{bmatrix} 5 & 2 \\ 2 & 5 \end{bmatrix},$$

and we are given that the eigenvectors are  $\begin{bmatrix} 1 & 1 \end{bmatrix}^T$  corresponding to the eigenvalue  $\lambda = 7$ , and  $\begin{bmatrix} 1 & -1 \end{bmatrix}^T$  corresponding to the eigenvalue  $\lambda = 3$ .

It follows that the general solution is

$$\begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} 1 \\ 1 \end{bmatrix} (C_1 e^{\sqrt{7}t} + C_2 e^{-\sqrt{7}t}) + \begin{bmatrix} 1 \\ -1 \end{bmatrix} (C_3 e^{\sqrt{3}t} + C_4 e^{-\sqrt{3}t}).$$

**Solution to Exercise 14**

Using the given eigenvalues and eigenvectors, we obtain the general solution

$$\begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix} (C_1 e^{\sqrt{2}t} + C_2 e^{-\sqrt{2}t}) + \begin{bmatrix} 1 \\ -5 \\ 0 \end{bmatrix} (C_3 \cos(\sqrt{3}t) + C_4 \sin(\sqrt{3}t)) \\ + \begin{bmatrix} -5 \\ 4 \\ 14 \end{bmatrix} (C_5 e^{2t} + C_6 e^{-2t}).$$

**Solution to Exercise 15**

In the solution to Example 11(b), we used a rather pedestrian approach to finding the particular solution. Let's do this a bit more smartly this time. First, write the general solution as

$$\mathbf{x}(t) = \mathbf{v}_1 (C_1 \cos(\sqrt{2}t) + C_2 \sin(\sqrt{2}t)) + \mathbf{v}_2 (C_3 \cos(2t) + C_4 \sin(2t)).$$

Setting  $t = 0$  and using the initial condition  $\mathbf{x}(0) = \mathbf{v}_2$  gives

$$\mathbf{x}(0) = \mathbf{v}_2 = C_1 \mathbf{v}_1 + C_3 \mathbf{v}_2.$$

Clearly this has solution  $C_1 = 0$  and  $C_3 = 1$ .

Now, differentiating the general solution with respect to  $t$  gives

$$\dot{\mathbf{x}}(t) = \sqrt{2} \mathbf{v}_1 (-C_1 \sin(\sqrt{2}t) + C_2 \cos(\sqrt{2}t)) \\ + 2 \mathbf{v}_2 (-C_3 \sin(2t) + C_4 \cos(2t)),$$

and using the initial condition  $\dot{\mathbf{x}}(0) = \mathbf{0}$  gives

$$\mathbf{0} = \sqrt{2} C_2 \mathbf{v}_1 + 2 C_4 \mathbf{v}_2.$$

This clearly has solution  $C_2 = C_4 = 0$ . Substituting these values in the general solution gives

$$\mathbf{x}(t) = \mathbf{v}_2 \cos(2t) = \begin{bmatrix} 1 \\ -1 \end{bmatrix} \cos(2t).$$

Clearly this satisfies the initial conditions.

**Solution to Exercise 16**

$\omega_2 > \omega_1$ , so  $\omega_2$  is out-of-phase and  $\omega_1$  is in-phase.

**Solution to Exercise 17**

Since the components of  $\mathbf{v}_2$  have the same sign, this must give rise to the in-phase mode.

**Solution to Exercise 18**

- (a) The matrix of coefficients has negative eigenvalues. The first eigenvalue gives the term

$$\mathbf{x}_1 = \begin{bmatrix} 1 \\ 1 \end{bmatrix} (C_1 \cos(\sqrt{3}t) + C_2 \sin(\sqrt{3}t)),$$

where  $C_1$  and  $C_2$  are arbitrary real constants. This is the in-phase normal mode solution. The second eigenvalue gives the term

$$\mathbf{x}_2 = \begin{bmatrix} 1 \\ -1 \end{bmatrix} (C_3 \cos(\sqrt{5}t) + C_4 \sin(\sqrt{5}t)),$$

where  $C_3$  and  $C_4$  are arbitrary real constants. This is the out-of-phase normal mode solution.

- (b) Following the same reasoning as in Example 11 and Exercise 15, these initial conditions give rise to the normal mode solution

$$\mathbf{x}(t) = \mathbf{v}_1 \cos(\sqrt{3}t) = \begin{bmatrix} 1 \\ 1 \end{bmatrix} \cos(\sqrt{3}t).$$

It is obvious that this solution satisfies the given initial conditions.

## Acknowledgements

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Figure 2: Taken from:

[http://en.wikipedia.org/wiki/File:Phillips\\_and\\_MONIAC\\_LSE.jpg](http://en.wikipedia.org/wiki/File:Phillips_and_MONIAC_LSE.jpg).

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